

# LOCAL AUTOMORPHISMS OF FINITARY INCIDENCE ALGEBRAS

JORDAN COURTEMANCHE, MANFRED DUGAS, AND DANIEL HERDEN

ABSTRACT. Let  $R$  be a commutative, indecomposable ring with identity and  $(P, \leq)$  a partially ordered set. Let  $FI(P)$  denote the finitary incidence algebra of  $(P, \leq)$  over  $R$ . We will show that, in most cases, local automorphisms of  $FI(P)$  are actually  $R$ -algebra automorphisms. In fact, the existence of local automorphisms which fail to be  $R$ -algebra automorphisms will depend on the chosen model of set theory and will require the existence of measurable cardinals. We will discuss local automorphisms of cartesian products as a special case in preparation of the general result.

## CONTENTS

1. Introduction	1
2. Finitary incidence algebras	3
3. Local automorphisms of cartesian products	5
3.1. Surjective local automorphisms	6
3.2. The exceptional case $ R  = 2$	7
3.3. Non-surjective local automorphisms	8
3.4. $R$ -algebra endomorphisms	14
4. Local automorphisms of finitary incidence algebras	16
4.1. Step 1: Splitting off $\widehat{\rho}$	17
4.2. Step 2: Splitting off $\psi_f$	19
4.3. Step 3: Splitting off $M_\sigma$	24
4.4. Step 4: Finish	25
References	28

## 1. INTRODUCTION

Let  $R$  be a commutative ring with identity and  $A$  some  $R$ -algebra. Let  $n$  denote a positive integer. The  $R$ -linear map  $\eta : A \rightarrow A$  is called an  *$n$ -local automorphism* if for any elements  $a_i \in A$ ,  $1 \leq i \leq n$ , there exists some  $R$ -algebra automorphism  $\varphi : A \rightarrow A$  such that  $\eta(a_i) = \varphi(a_i)$  for all  $1 \leq i \leq n$ . Any 1-local automorphism is simply called a *local automorphism*. Local automorphisms of algebras have attracted attention over the years. For example, in 1990, Larson and Sourour [9]

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Corresponding author: Manfred Dugas.

proved that if  $X$  is an infinite-dimensional Banach space and  $\eta$  is a local automorphism of  $\mathfrak{B}(X)$ , the algebra of all bounded linear operators on  $X$ , then  $\eta$  is an automorphism of  $\mathfrak{B}(X)$ . In 1999, Crist [3] studied the local automorphisms of finite dimensional CSL algebras. In 2003, Hadwin and Li [5] showed that any surjective 2-local automorphism of a nest algebra  $\mathcal{A}$  is an automorphism. We refer to the introductions and references of these three papers for more details on the history and results regarding local automorphisms of algebras. For further reading, see also [1, 2, 11].

From now on, we will always assume that the commutative ring  $R$  is indecomposable, i.e., 0 and 1 are the only idempotent elements of  $R$ . Let  $(P, \leq)$  be a poset. Generalizing the notion of the incidence algebra [10, 14] of a locally finite poset  $(P, \leq)$  over  $R$ , Khripchenko and Novikov [8] introduced the concept of the *finitary incidence algebra*  $FI(P)$  for arbitrary posets  $(P, \leq)$  in 2009. One year later, Khripchenko showed that the algebra  $FI(P)$  has exactly the same kind of automorphisms as in the case of a locally finite poset  $(P, \leq)$ , cf. [7, 13]. Our goal is to determine the local automorphisms of  $FI(P)$ . The proof will introduce a number of new ideas and concepts for finitary incidence algebras, combining methods from set theory, combinatorics and algebra.

We will associate a cardinal  $\mu_R$  to the ring  $R$  as follows: If  $R$  is finite, then  $\mu_R = \aleph_0$ . If  $R$  is infinite, then  $\mu_R$  is the least cardinal that allows a non-trivial  $|R|^+$ -additive measure with values 0 and 1. Let  $\text{Aut}(A)$  denote the group of all  $R$ -algebra automorphisms of the  $R$ -algebra  $A$ , which is a subset of  $\text{LAut}(A)$ , the set of all local automorphisms of  $A$ . Our main result is the following:

**Main Theorem 1.1.** *Let  $(P, \leq)$  be a poset and  $R$  an indecomposable ring.*

- (a) *If  $|P| < \mu_R$ , then  $\text{LAut}(FI(P)) = \text{Aut}(FI(P))$ .*
- (b) *If  $|P| \geq \mu_R$ , then non-surjective local automorphisms may exist, and examples of local automorphisms which are not  $R$ -algebra automorphisms are provided.*
- (c) *If  $\eta \in \text{LAut}(FI(P))$  is surjective and  $|R| \geq 3$ , then  $\eta \in \text{Aut}(FI(P))$ .*

Let  $\mu_0$  denote the least measurable cardinal, if it exists. Measurable cardinals are large cardinals known to not exist in many set-theoretic models of ZFC, for example  $V = L$ , and it cannot be proven that the existence of measurable cardinals is consistent with ZFC. We refer to [6] for more details on measurable cardinal. For  $\aleph_0 \leq |R| < \mu_0$ , we have  $\mu_R = \mu_0$ , and we note the following immediate consequence of Theorem 1.1(a).

**Corollary 1.2.** *Let  $R$  be an infinite indecomposable ring and  $(P, \leq)$  a poset such that  $|P| < \mu_0$ . Then  $\text{LAut}(FI(P)) = \text{Aut}(FI(P))$ .*

If  $(P, \leq)$  is a trivial poset, i.e., the partial order is just equality, then  $\Pi = FI(P) = \prod_{x \in P} Re_x$  is just the full cartesian product of  $|P|$  copies of  $R$ . Note that for any poset  $(P, \leq)$ , the  $R$ -algebra  $\Pi$  is naturally an epimorphic image of  $FI(P)$ . This motivates the study of  $\text{Aut}(\Pi)$  and  $\text{LAut}(\Pi)$  as preparation for a proof of Theorem 1.1.

For any permutation  $\rho : P \rightarrow P$ , the map  $\hat{\rho} : \Pi \rightarrow \Pi$  with  $\hat{\rho}(\sum_{x \in P} r_x e_x) = \sum_{x \in P} r_x e_{\rho(x)}$  is in  $\text{Aut}(\Pi)$ . Our starting point will be the simple observation that every  $R$ -algebra automorphisms of  $\Pi$  arises like this from a suitable permutation  $\rho$  of  $P$ .

**Proposition 3.2.**  $\text{Aut}(\Pi) = \{\hat{\rho} : \rho \text{ is a permutation of } P\}$ .

From here we set out to discuss local automorphisms of  $\Pi$  and their properties. By nature, local automorphisms will be very close to  $R$ -algebra automorphisms, and the question whether, or when,  $\text{Aut}(\Pi) = \text{LAut}(\Pi)$  holds will be pursued systematically in Section 3. For 2-local automorphisms we will show the following.

**Corollary 3.9.** *If  $\eta \in \text{LAut}(\Pi)$  is a surjective 2-local automorphism, then  $\eta \in \text{Aut}(\Pi)$ .*

In addition, we have a correlating result for local automorphisms.

**Theorem 3.6.** *Let  $|R| \geq 3$  and  $\eta \in \text{LAut}(\Pi)$  such that  $e_x \in \text{Im}(\eta)$ , the image of  $\eta$ , for all  $x \in P$ . Then  $\eta \in \text{Aut}(\Pi)$ .*

Thus, under some weak assumption of surjectivity, local automorphisms indeed coincide with  $R$ -algebra automorphisms. The case  $|R| = 2$ , i.e.,  $R = \mathbb{Z}_2$ , however, is special.

**Theorem 3.11.** *Let  $R = \mathbb{Z}_2$  and  $\kappa = |P|$  an infinite cardinal. Then there exist  $2^{2^\kappa}$ -many non-surjective local automorphisms of  $\Pi$ , as well as  $2^{2^\kappa}$ -many bijective local automorphisms of  $\Pi$  that **are not**  $R$ -algebra automorphisms of  $\Pi$ .*

Our next main result shows that “ $n$ -local” does, in general, not imply “surjective” for local automorphisms:

**Lemma 3.13.** *Let  $R$  be finite and  $P$  be countably infinite. Then there exists some  $\eta \in \text{LAut}(\Pi)$  such that  $\eta$  is an  $n$ -local automorphism for all natural numbers  $n$ , but  $\eta$  is **not** surjective. In particular,  $\eta \notin \text{Aut}(\Pi)$ .*

This result holds independently of the chosen model of ZFC and generalizes to infinite rings  $R$  as follows.

**Main Theorem 1.3.**

- (a)  $\text{LAut}(\Pi) = \text{Aut}(\Pi)$  if and only if  $|P| < \mu_R$ . Moreover,
- (b) Let  $|P| \geq \mu_R$ . Then there exists some unital  $R$ -linear map  $\eta$  such that  $\eta$  is an  $n$ -local automorphism for all natural numbers  $n$ , but  $\eta$  is **not** surjective. In particular,  $\eta \notin \text{Aut}(\Pi)$ .

This theorem is notable, as it links the existence of nontrivial local automorphisms  $\eta \notin \text{Aut}(\Pi)$  to the existence of measurable cardinals. Therefore, the existence of nontrivial local automorphisms depends on the chosen model of set theory.

Theorem 1.3 extends to  $\text{End}(\Pi)$ , the set of  $R$ -algebra endomorphisms of  $\Pi$ . For a subset  $I$  of  $P$ , we define  $e_I = \sum_{x \in I} e_x$ , and we call  $\eta \in \text{End}(\Pi)$  *induced* if there is a family  $\{A_x : x \in P\}$  of pairwise disjoint subsets of  $P$ , such that  $\eta(\sum_{x \in P} a_x e_x) = \sum_{x \in P} a_x e_{A_x}$  for all  $\sum_{x \in P} a_x e_x \in \Pi$ . We have the following result.

**Theorem 3.28.** *Let  $R$  be a field. Then every  $\eta \in \text{End}(\Pi)$  is induced if and only if  $|P| < \mu_R$ .*

Theorem 1.3 will serve as a stepping stone for proving Theorem 1.1 in Section 4. With Section 2 we include a short overview of finitary incidence algebras. Our notation is standard, cf. [12].

## 2. FINITARY INCIDENCE ALGEBRAS

For the convenience of the reader, we recall some definitions and results. Let  $(P, \leq)$  denote a poset and  $R$  a commutative ring with 1.

**Definition 2.1** (Khripchenko, Novikov, [8]).

- (a) We call  $I(P) = \prod_{x,y \in P, x \leq y} Re_{xy}$  the induced incidence space of  $(P, \leq)$  over  $R$ .  $I(P)$  is an  $R$ -module under componentwise addition and scalar multiplication.
- (b) Let

$$FI(P) = \left\{ a = \sum_{x \leq y} a_{xy} e_{xy} \in I(P) : \right.$$

$$\left. \{(u, v) : x \leq u < v \leq y \text{ and } a_{uv} \neq 0\} \text{ is finite for all } x < y \right\}.$$

Then  $FI(P)$  is an  $R$ -algebra, the induced finitary incidence algebra of  $(P, \leq)$  over  $R$ . The multiplication (of formal sums) is induced by

$$e_{xu} e_{vy} = \begin{cases} e_{xy}, & \text{if } x \leq u = v \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

Note that  $I(P)$  is a bimodule over the ring  $FI(P)$  and  $a = \sum_{x \leq y} a_{xy} e_{xy} \in FI(P)$  is a unit in  $FI(P)$  if and only if all coefficients  $a_{xx}$  are units in  $R$ . Moreover,

$$Z := Z(FI(P)) = \{a = \sum_{x \leq y} a_{xy} e_{xy} \in FI(P) : a_{xx} = 0 \text{ for all } x \in P\}$$

is a two-sided ideal of  $FI(P)$  and  $FI(P)/Z \cong \prod_{x \in P} Re_{xx}$ .

**Definition 2.2.**

- (a) For any invertible  $f \in FI(P)$  let  $\psi_f$  be the induced inner automorphism, and  $\text{Inn}(FI(P))$  denotes the set of all inner automorphisms of  $FI(P)$ . Note that  $\text{Inn}(FI(P))$  is a normal subgroup of  $\text{Aut}(FI(P))$ .
- (b) For any order automorphism  $\rho$  of  $(P, \leq)$  let  $\hat{\rho}$  denote the induced  $R$ -algebra automorphism of  $FI(P)$  with

$$\hat{\rho} \left( \sum_{x \leq y} a_{xy} e_{xy} \right) = \sum_{x \leq y} a_{xy} e_{\rho(x)\rho(y)}.$$

Let  $\text{Aut}(P) = \{\hat{\rho} : \rho \text{ order automorphism of } P\}$ , a subgroup of  $\text{Aut}(FI(P))$ .

- (c) Given units  $\sigma_{xy} \in R$  with  $\sigma_{xy}\sigma_{yz} = \sigma_{xz}$  for all  $x \leq y \leq z$  let  $M_\sigma$  denote the induced  $R$ -algebra automorphism (Schur multiplication) of  $FI(P)$ ,

$$M_\sigma \left( \sum_{x \leq y} a_{xy} e_{xy} \right) = \sum_{x \leq y} a_{xy} \sigma_{xy} e_{xy}.$$

Let  $\text{Mult}(FI(P))$  denote the set of all Schur multiplications of  $FI(P)$ . Then  $\text{Mult}(FI(P))$  is a subgroup of  $\text{Aut}(FI(P))$ .

Quite frequently, we will apply the following result:

**Theorem 2.3** (Khripchenko, [7]). Let  $(P, \leq)$  be a poset and  $R$  a commutative ring with 1. If  $R$  is indecomposable, then every  $\varphi \in \text{Aut}(FI(P))$  decomposes canonically:

$$\varphi = \psi_f \circ M_\sigma \circ \hat{\rho}.$$

For the original result on incidence algebras, cf. [12, 13].

The theorem implies that  $\text{Aut}(FI(P))$  is a product of the subgroups  $\text{Inn}(FI(P))$ ,  $\text{Mult}(FI(P))$  and  $\text{Aut}(P)$ . It follows from the definition, that  $\text{Aut}(P)$  normalizes

$\text{Mult}(FI(P))$ , which implies that  $\text{Aut}(FI(P))$  is the product of these three subgroups in any order of the factors. Note that, in general,

$$\text{Inn}(FI(P)) \cap \text{Mult}(FI(P)) \neq \{\text{id}_{FI(P)}\},$$

but  $\text{Aut}(FI(P))$  is a semidirect product of  $\text{Inn}(FI(P)) \text{Mult}(FI(P))$  by  $\text{Aut}(P)$ . Moreover, the subgroup

$$\text{Inn}(FI(P)) \text{Mult}(FI(P)) \subseteq \text{Aut}(FI(P))$$

corresponds to those  $R$ -algebra automorphisms of  $FI(P)$  which induce the identity map on  $FI(P)/Z$ . One of the many consequences is that the ideal  $Z$  is invariant under all  $R$ -algebra automorphisms of  $FI(P)$ .

Let  $D(P) = FI(P) \oplus I(P)$  denote the idealization (Dorroh extension) of  $I(P)$  by  $FI(P)$ . A complete description of  $\text{Aut}(D(P))$  was obtained in [4].

### 3. LOCAL AUTOMORPHISMS OF CARTESIAN PRODUCTS

In Section 3 we will investigate the structure of local automorphisms and  $R$ -algebra endomorphisms of cartesian products. We fix the following notations and definitions.

**Notation 3.1.** Let  $P$  be a set and  $R$  a commutative, indecomposable ring, i.e., 0 and 1 are the only idempotent elements of the ring  $R$ . Let  $\Pi = \prod_{x \in P} Re_x$  be the cartesian product. Then  $\Pi$  is an  $R$ -algebra, and by  $\text{End}(\Pi)$  and  $\text{Aut}(\Pi)$  we denote the set of all  $R$ -algebra endomorphisms and automorphisms, respectively, of  $\Pi$ . We will associate any element  $a \in \Pi$  with its standard representation  $a = \sum_{x \in P} a_x e_x$ . For  $X \subseteq P$  define  $e_X = \sum_{x \in X} e_x$ . Let  $\rho : P \rightarrow P$  be any map. Define  $\hat{\rho} : \Pi \rightarrow \Pi$  by  $\hat{\rho}(\sum_{x \in P} a_x e_x) = \sum_{x \in P} a_x e_{\rho(x)}$  for all  $a \in \Pi$ . It is readily verified that  $\hat{\rho}$  is an  $R$ -algebra endomorphism of  $\Pi$  if and only if  $\rho$  is injective. Note that  $\hat{\rho}$  may not be unital.

We record a first easy observation on  $\hat{\rho}$ .

**Proposition 3.2.** Let  $R$  be an indecomposable ring. Then for  $\Pi = \prod_{x \in P} Re_x$  holds

$$\text{Aut}(\Pi) = \{\hat{\rho} : \rho \text{ is a permutation of } P\}.$$

*Proof.* If  $\rho$  is a permutation of  $P$ , then  $\widehat{\rho^{-1}} = \hat{\rho}^{-1}$ , and thus  $\hat{\rho}$  is an automorphism. Now assume that  $\eta \in \text{Aut}(\Pi)$  is an automorphism and let  $E = \{e_x : x \in P\}$ . Then  $E$  is the set of all primitive idempotents of  $\Pi$  and thus  $\eta(E) \subseteq E$  as well as  $\eta^{-1}(E) \subseteq E$ . This shows that  $\eta|_E : E \rightarrow E$  is bijective and we infer that there exists a permutation  $\rho$  of  $P$  such that  $\eta(e_x) = e_{\rho(x)}$  for all  $x \in P$ . Let  $a = \sum_{x \in P} a_x e_x \in \Pi$ . Then

$$\eta(a)e_{\rho(x)} = \eta(a)\eta(e_x) = \eta(ae_x) = \eta(a_x e_x) = a_x \eta(e_x) = a_x e_{\rho(x)}$$

for all  $x \in P$ . Thus  $a_x$  is the entry of the  $e_{\rho(x)}$ -coordinate of  $\eta(a)$ . This shows that  $\eta = \hat{\rho}$ .  $\square$

We continue with the central definition of this section.

**Definition 3.3.**

- (a) An  $R$ -linear map  $\eta : FI(P) \rightarrow FI(P)$  is called an  $n$ -local automorphism if for every choice of  $a_i \in FI(P)$ ,  $1 \leq i \leq n$ , there exists some  $R$ -algebra automorphism  $\varphi \in \text{Aut}(FI(P))$  with  $\eta(a_i) = \varphi(a_i)$  for all  $1 \leq i \leq n$ .

- (b) We will refer to 1-local automorphisms simply as local automorphisms. With  $\text{LAut}(FI(P))$  we denote the set of local automorphisms.

**Remark 3.4.** Every  $R$ -algebra automorphism is an  $n$ -local automorphism for all  $n > 0$ . However, a local automorphism does not need to be an  $R$ -algebra homomorphism. Clearly,  $n$ -local automorphisms will preserve multiplication for  $n \geq 3$ .

We note some general properties of local automorphisms.

**Proposition 3.5.** Let  $\eta$  be a local automorphism of an  $R$ -algebra  $A$ . Then the following holds:

- (a)  $\eta$  is injective.
- (b)  $\eta$  preserves idempotents.
- (c)  $\eta$  preserves primitive idempotents.

*Proof.* Let  $a \in \text{Ker}(\eta)$ . Then  $a \in \text{Ker}(\varphi)$  for some  $\varphi \in \text{Aut}(A)$ . Thus  $\text{Ker}(\eta) = 0$ , and (a) holds. Similarly, as  $R$ -algebra automorphisms preserve idempotents and primitive idempotents, (b) and (c) hold.  $\square$

**3.1. Surjective local automorphisms.** In this section we present some first results on surjective local automorphisms. We start with the main result.

**Theorem 3.6.** Let  $R$  be an indecomposable ring with  $|R| \geq 3$ , and  $\eta$  a local automorphism of  $\Pi = \prod_{x \in P} Re_x$  such that  $e_x \in \text{Im}(\eta)$  for all  $x \in P$ . Then  $\eta$  is an  $R$ -algebra automorphism, and  $\eta = \hat{\rho}$  for some permutation  $\rho : P \rightarrow P$ .

*Proof.* Let  $E = \{e_x : x \in P\}$  be the set of all primitive idempotents of  $\Pi$ . Then  $\eta(E) \subseteq E$ , and  $E$  is closed under preimages under  $\eta$ . We infer that  $\eta(e_x) = e_{\rho(x)}$  defines a permutation  $\rho : P \rightarrow P$ .

Fix some  $y \in P$  and consider an element  $a = \sum_{x \in P} a_x e_x \in \Pi$  such that

$$(3.1) \quad a_x \neq a_y \quad \text{for all } y \neq x \in P.$$

Since  $\eta$  is a local automorphism of  $\Pi$ , there exist permutations  $\sigma, \tau : P \rightarrow P$  such that

$$\eta(a) = \hat{\sigma}(a) = \sum_{x \in P} a_x e_{\sigma(x)} \quad \text{and} \quad \eta(a + e_y) = \hat{\tau}(a + e_y) = (a_y + 1)e_{\tau(y)} + \sum_{y \neq x \in P} a_x e_{\tau(x)}.$$

In particular,

$$(a_y + 1)e_{\tau(y)} + \sum_{y \neq x \in P} a_x e_{\tau(x)} = \eta(a + e_y) = \eta(a) + \eta(e_y) = e_{\rho(y)} + \sum_{x \in P} a_x e_{\sigma(x)}.$$

For  $\rho(y) \neq \sigma(y)$ , note that  $a_y$  is the  $e_{\sigma(y)}$ -coordinate of the right-hand side, while all coordinates on the left-hand side differ from  $a_y$ . Thus, we infer  $\rho(y) = \sigma(y)$  or, in terms of coordinate entries of  $a$  and  $\eta(a)$ ,

$$(3.2) \quad (\eta(a))_{\rho(y)} = (\eta(a))_{\sigma(y)} = a_y.$$

Now let  $a \in \Pi$  be arbitrary. If we can decompose  $a = b + c$  in such a way that  $b, c \in \Pi$  both satisfy Property (3.1), then (3.2) applies to  $b$  and  $c$ , and

$$(3.3) \quad (\eta(a))_{\rho(y)} = (\eta(b))_{\rho(y)} + (\eta(c))_{\rho(y)} = b_y + c_y = a_y = (\hat{\rho}(a))_{\rho(y)}.$$

We define  $b \in \Pi$  by choosing  $b_y = 0$  and  $b_x \in R \setminus \{0, a_x - a_y\}$  for all  $y \neq x \in P$ . Let  $c = a - b$ . Then  $c_y = a_y$  and

$$c_x = a_x - b_x \notin \{a_x - 0, a_x - (a_x - a_y)\} = \{a_x, a_y\} = \{a_x, c_y\}.$$

for all  $y \neq x \in P$ . Thus, Property (3.1) holds for both  $b$  and  $c$ .

Note that (3.3) holds for all  $y \in P$ , thus  $\eta(a) = \hat{\rho}(a)$  for all  $a \in \Pi$ , and  $\eta = \hat{\rho}$ .  $\square$

We note an important immediate consequence of this theorem.

**Corollary 3.7.** *Let  $R$  be an indecomposable ring with  $|R| \geq 3$ , and  $\eta$  a surjective local automorphism of  $\Pi = \prod_{x \in P} Re_x$ . Then  $\eta$  is an  $R$ -algebra automorphism.*

Theorem 3.6 and Corollary 3.7 fail in the case of  $|R| = 2$  as will be detailed in Section 3.2. In this respect, the 2-local automorphisms of  $\Pi$  are better behaved.

**Theorem 3.8.** *Let  $R$  be an indecomposable ring, and  $\eta$  a 2-local automorphism of  $\Pi = \prod_{x \in P} Re_x$  such that  $e_x \in \text{Im}(\eta)$  for all  $x \in P$ . Then  $\eta$  is an  $R$ -algebra automorphism, and  $\eta = \hat{\rho}$  for some permutation  $\rho : P \rightarrow P$ .*

*Proof.* As seen in Theorem 3.6,  $\eta$  induces a permutation  $\rho : P \rightarrow P$  with  $\eta(e_x) = e_{\rho(x)}$  for all  $x \in P$ . Let  $a = \sum_{x \in P} a_x e_x \in \Pi$  and  $y \in P$ . Since  $\eta$  is a 2-local automorphism, there exists a permutation  $\sigma : P \rightarrow P$  such that  $\eta(a) = \hat{\sigma}(a)$  and  $\eta(e_y) = \hat{\sigma}(e_y)$ . It follows that  $e_{\rho(y)} = \eta(e_y) = \hat{\sigma}(e_y) = e_{\sigma(y)}$ , and thus  $\rho(y) = \sigma(y)$ . Now we have

$$(3.4) \quad (\eta(a))_{\rho(y)} = (\eta(a))_{\sigma(y)} = (\hat{\sigma}(a))_{\sigma(y)} = a_y = (\hat{\rho}(a))_{\rho(y)}.$$

Note that (3.4) holds for all  $y \in P$ , thus  $\eta(a) = \hat{\rho}(a)$  for all  $a \in \Pi$ , and  $\eta = \hat{\rho}$ .  $\square$

We mention the following counterpart to Corollary 3.7.

**Corollary 3.9.** *Let  $R$  be an indecomposable ring, and  $\eta$  a surjective 2-local automorphism of  $\Pi = \prod_{x \in P} Re_x$ . Then  $\eta$  is an  $R$ -algebra automorphism.*

**3.2. The exceptional case  $|R| = 2$ .** Now we consider the special case  $|R| = 2$ , which was left out in Theorem 3.6. In this case,  $R = \mathbb{Z}_2 = \{0, 1\}$  is the field with two elements, and the  $R$ -algebra  $\Pi = (\Pi, +, \cdot)$  is naturally isomorphic to  $\mathfrak{B} = (\mathcal{P}(P), \Delta, \cap)$ , where  $\mathcal{P}(P)$  is the power set of  $P$ , and  $\Delta$  denotes the symmetric difference. Let  $\varphi$  be an automorphism of  $\mathfrak{B}$ . By Proposition 3.2 there exists some permutation  $\rho : P \rightarrow P$  such that  $\varphi = \hat{\rho}$ , i.e.,

$$\varphi(X) = \{\rho(x) : x \in X\} = \rho(X) \quad \text{and} \quad P \setminus \varphi(X) = \{\rho(x) : x \in P \setminus X\} = \rho(P \setminus X)$$

for all subsets  $X$  of  $P$ . Note, in particular, that  $\varphi$  preserves cardinalities,

$$|\varphi(X)| = |X| \quad \text{and} \quad |P \setminus \varphi(X)| = |P \setminus X| \quad \text{for all } X \subseteq P.$$

This observation motivates the following characterization of local automorphisms.

**Proposition 3.10.** *Let  $\eta : \mathfrak{B} \rightarrow \mathfrak{B}$  be a map. Then  $\eta$  is a local automorphism of  $\mathfrak{B}$  if and only if*

$$(3.5) \quad |\eta(X)| = |X| \quad \text{and} \quad |P \setminus \eta(X)| = |P \setminus X| \quad \text{for all } X \subseteq P.$$

*Proof.* Assume that  $\eta : \mathfrak{B} \rightarrow \mathfrak{B}$  is a local automorphism of  $\mathfrak{B}$ , and  $X \subseteq P$ . Then there exists an automorphism  $\varphi : \mathfrak{B} \rightarrow \mathfrak{B}$  with  $\eta(X) = \varphi(X)$ , which yields  $|\eta(X)| = |\varphi(X)| = |X|$  and  $|P \setminus \eta(X)| = |P \setminus \varphi(X)| = |P \setminus X|$ .

Now assume that  $\eta : \mathfrak{B} \rightarrow \mathfrak{B}$  is a map with  $|\eta(X)| = |X|$  and  $|P \setminus \eta(X)| = |P \setminus X|$  for all  $X \subseteq P$ . Thus, for any given  $X \subseteq P$  we can choose some permutation  $\rho : P \rightarrow P$  with  $\rho(X) = \eta(X)$ . Then  $\hat{\rho}$  is an automorphism of  $\mathfrak{B}$  such that  $\eta(X) = \hat{\rho}(X)$ , and  $\eta$  is a local automorphism.  $\square$



Property (3.5) provides a simple and powerful tool for constructing various local automorphisms. We will use it to give counterexamples for Theorem 3.6 and Corollary 3.7, namely, bijective local automorphisms of  $\mathfrak{B}$  which are no automorphisms.

**Theorem 3.11.** *Let  $\kappa = |P|$  be infinite. Then there exist  $2^{2^\kappa}$ -many non-surjective local automorphisms of  $\mathfrak{B}$ , as well as  $2^{2^\kappa}$ -many bijective local automorphisms of  $\mathfrak{B}$  that are not  $R$ -algebra automorphisms of  $\mathfrak{B}$ .*

*Proof.* Let

$$K = \{X \subseteq P : |X| < \kappa \text{ or } |P \setminus X| < \kappa\}.$$

It is easy to see that  $K$  is closed with respect to  $\Delta$  and thus a subgroup of cardinality  $|K| = 2^{<\kappa}$  of the elementary abelian 2-group  $\mathfrak{B} = (\mathcal{P}(P), \Delta)$ . Thus, there is a subgroup  $C$  of  $\mathfrak{B}$  such that  $\mathfrak{B} = K \oplus C$  with respect to  $\Delta$ . We claim that

$$|C| = |\mathfrak{B}| = 2^\kappa.$$

To see this, let us call a family  $\mathcal{F}$  of subsets of  $P$  *independent* if for any distinct sets  $X_1, \dots, X_n, Y_1, \dots, Y_m$  in  $\mathcal{F}$  the intersection

$$\bigcap_{i=1}^n X_i \cap \bigcap_{j=1}^m (P \setminus Y_j)$$

has cardinality  $\kappa$ . With [6, Lemma 7.7] there exists an independent family  $\mathcal{F} \subseteq \mathcal{P}(P)$  of cardinality  $|\mathcal{F}| = 2^\kappa$ , and it is easy to check that  $\mathcal{F}$  is a set of independent elements modulo  $K$ . Hence  $|C| = |\mathcal{F}| = 2^\kappa$ .

Now let  $\theta : C \rightarrow C$  be any injective  $\Delta$ -homomorphism, and define  $\eta : \mathfrak{B} \rightarrow \mathfrak{B}$  by  $\eta = \text{id}_K \oplus \theta$ . Then  $\eta$  is a homomorphism with respect to  $\Delta$  and with (3.5) a local automorphism of  $\mathfrak{B}$ .

If  $\theta$  is chosen to be non-surjective, then  $\eta$  is a non-surjective local automorphism, and there are  $2^{2^\kappa}$  such maps. Of course, if  $\theta$  is chosen to be bijective, then  $\eta$  is a bijective local automorphism, and there are again  $2^{2^\kappa}$  such maps. Note, however, that  $\mathfrak{B}$  has only  $2^\kappa$  many automorphisms since those are induced by permutations  $\rho : P \rightarrow P$ , cf. Proposition 3.2.  $\square$

In the last theorem we caught a first glimpse of non-surjective local automorphisms. These will become the topic of Section 3.3. Note also, that independent families, as introduced in the proof of Theorem 3.11, closely relate to ultrafilters. This connection will intensify in the following section.

**3.3. Non-surjective local automorphisms.** As seen in Section 3.1, surjective local automorphisms basically coincide with  $R$ -algebra automorphisms. In this section, we want to discuss the possible existence of non-surjective local automorphisms on  $\Pi$ . We will see that this problem relates to a specific large cardinal number  $\mu_R$ , cf. Definition 3.16.

We will start discussing the central ultrafilter construction for the special case of finite indecomposable rings  $R$ . This will need a small auxiliary result.

**Lemma 3.12.** *Let  $\omega = X \dot{\cup} Y$  be a partition of  $\omega$  such that  $Y$  is infinite, and let  $\sigma : \omega \rightarrow \omega \setminus \{0\}$  be a bijection. Then there exists a permutation  $\rho$  of  $\omega$  such that  $\rho \upharpoonright_X = \sigma \upharpoonright_X$  and  $\rho(Y) = \sigma(Y) \dot{\cup} \{0\}$ .*



*Proof.* Note that  $\omega = \sigma(X) \dot{\cup} \sigma(Y) \dot{\cup} \{0\}$  is a partition of  $\omega$ . Let  $Y = \{y_i : i \in \omega\}$  be an enumeration of the elements of  $Y$  and define

$$\rho(x) = \begin{cases} \sigma(x) & \text{for } x \in X, \\ 0 & \text{for } x = y_0, \\ \sigma(y_{i-1}) & \text{for } x = y_i, 1 \leq i \in \omega. \end{cases}$$

Note that  $\rho(X) = \sigma(X)$  and  $\rho(Y) = \sigma(Y) \cup \{0\}$ , and thus  $\rho$  is surjective. Moreover,  $\rho \upharpoonright X$  and  $\rho \upharpoonright Y$  are injective with

$$\rho(X) \cap \rho(Y) = \sigma(X) \cap (\sigma(Y) \cup \{0\}) = \emptyset,$$

which shows that  $\rho$  is bijective.  $\square$

With this we are ready for our first ultrafilter construction of a non-surjective local automorphism.

**Lemma 3.13.** *Let  $R$  be a finite indecomposable ring and  $\Pi = \prod_{i \in \omega} Re_i$ . Then there exists some unital  $\eta \in \text{End}(\Pi)$  such that  $\eta$  is an  $n$ -local automorphism for all  $n > 0$ , but not surjective. In particular,  $\eta$  is not an  $R$ -algebra automorphism of  $\Pi$ .*

*Proof.* Let  $\mathcal{U}$  be a non-principal ultrafilter for the set  $\omega$  that contains all cofinite subsets of  $\omega$ . Let  $a \in \Pi$ . Then there exist disjoint sets  $[a]_r \subseteq \omega$ ,  $r \in R$ , such that  $a = \sum_{r \in R} re_{[a]_r}$ . Since  $\mathcal{U}$  is an ultrafilter, there exists exactly one  $s =: \varphi(a) \in R$  such that  $u(a) := [a]_s \in \mathcal{U}$ . It is easy to check that  $\varphi : \Pi \rightarrow R$  is an  $R$ -algebra homomorphism. Now define  $\eta : \Pi \rightarrow \Pi$  by

$$\eta(a) = \varphi(a)e_0 + \sum_{i \in \omega} a_i e_{i+1}$$

for all  $a = \sum_{i \in \omega} a_i e_i \in \Pi$ . Then  $\eta$  is an injective unital  $R$ -algebra homomorphism, but not surjective since, for example,  $e_0$  has no preimage under  $\eta$ .

Now let  $a \in \Pi$ . Note that  $u(a) \in \mathcal{U}$  is an infinite set. Let  $Y$  be any infinite subset of  $u(a)$ , and let  $X = \omega \setminus Y$ . Apply Lemma 3.12 to  $\omega = X \dot{\cup} Y$  and the *shift map*  $\sigma$  on  $\omega$ , where  $\sigma(i) = i + 1$  for all  $i \in \omega$ . This yields a permutation  $\rho$  of  $\omega$  with associated  $R$ -algebra automorphism  $\hat{\rho} \in \text{Aut}(\Pi)$ . Note that

$$\begin{aligned} \hat{\rho}(a) &= \hat{\rho} \left( \sum_{x \in X} a_x e_x + \sum_{y \in Y} a_y e_y \right) = \hat{\rho} \left( \sum_{x \in X} a_x e_x \right) + \hat{\rho} \left( \sum_{y \in Y} \varphi(a) e_y \right) \\ &= \hat{\rho} \left( \sum_{x \in X} a_x e_x \right) + \hat{\rho}(\varphi(a) e_Y) = \sum_{x \in X} a_x e_{\rho(x)} + \varphi(a) e_{\rho(Y)} \\ &= \sum_{x \in X} a_x e_{\sigma(x)} + \varphi(a) e_{\sigma(Y) \cup \{0\}} = \varphi(a) e_0 + \left( \sum_{x \in X} a_x e_{\sigma(x)} + \varphi(a) e_{\sigma(Y)} \right) \\ &= \varphi(a) e_0 + \hat{\sigma}(a) = \eta(a). \end{aligned}$$

This shows that  $\eta$  is a local automorphism.

Now let  $a_1, a_2, \dots, a_n \in \Pi$  and consider the infinite set  $Y = \bigcap_{1 \leq i \leq n} u(a_i) \in \mathcal{U}$ . Note that the automorphism  $\hat{\rho}$  constructed above will have the property that  $\eta(a_i) = \hat{\rho}(a_i)$  for all  $1 \leq i \leq n$ . Thus  $\eta$  is an  $n$ -local automorphism.  $\square$

We include the following immediate consequence of Lemma 3.13.

**Corollary 3.14.** *If  $R$  is a finite indecomposable ring and  $P$  an infinite set, then there exists an injective unital  $R$ -algebra endomorphism  $\eta$  of  $\Pi = \prod_{x \in P} Re_x$  such that  $\eta \neq \widehat{\rho}$  for all injective maps  $\rho : P \rightarrow P$ .*

*Proof.* Let  $R$  be finite,  $P$  infinite, and  $\{x_\alpha : \alpha < |P|\}$  an enumeration of  $P$ . Let the map  $\varphi : \Pi \rightarrow R$  be defined via an ultrafilter on  $P$  as in Lemma 3.13, and define

$$\eta(a) = \varphi(a)e_{x_0} + \sum_{\alpha < |P|} a_{x_\alpha} e_{x_{\alpha+1}}$$

for all  $a = \sum_{x \in P} a_x e_x \in \Pi$ . Then  $\eta$  is an injective unital  $R$ -algebra endomorphism of  $\Pi$ . Moreover,  $e_{x_0} \notin \eta(\Pi)$ , but  $\eta$  followed by the natural projection onto the  $e_{x_0}$ -coordinate is non-zero. This shows that  $\eta \neq \widehat{\rho}$  for all maps  $\rho : P \rightarrow P$ .  $\square$

The following example illustrates an interesting variation of the construction provided in the proof of Lemma 3.13.

**Example 3.15.** *Let  $R$  be a commutative ring with identity and consider the  $R$ -subalgebra*

$$A = \left\langle \sum_{i \in \omega} e_i, \bigoplus_{i \in \omega} Re_i \right\rangle$$

of  $\Pi = \prod_{i \in \omega} Re_i$ . Then each  $a \in A$  has the form

$$a = \sum_{i=0}^{n(a)} s_i e_i + \sum_{i \in \omega} a^* e_i$$

for  $s_i \in R$  and a unique  $a^* \in R$ . Now define an  $R$ -linear map  $\eta : A \rightarrow A$  by

$$\eta(a) = a^* e_0 + \sum_{i=0}^{n(a)} s_i e_{i+1} + \sum_{i \in \omega} a^* e_{i+1}$$

for all  $a \in A$ . Note that  $e_0$  is not in the image of  $\eta$ , and thus  $\eta$  is not surjective. Let  $n > 0$ . We will show that  $\eta$  is an  $n$ -local automorphism of  $A$ . To this end, let  $a_j \in A$  for  $1 \leq j \leq n$  and pick some integer  $m$  with  $m > n(a_j)$  for all  $1 \leq j \leq n$ . Define a permutation  $\rho : \omega \rightarrow \omega$  by  $\rho(i) = i + 1$  for all  $m \neq i < \omega$  and  $\rho(m) = 0$ . Then  $\rho$  induces an  $R$ -algebra automorphism  $\widehat{\rho}$  of the  $R$ -algebra  $A$  and it is easy to check that  $\eta(a_j) = \widehat{\rho}(a_j)$  for all  $1 \leq j \leq n$ , which shows that  $\eta$  is an  $n$ -local automorphism of  $A$  which is not surjective and thus not an  $R$ -algebra automorphism of  $A$ .

For fields  $R$ , we note another quite remarkable property of the local automorphism  $\eta$ : Let  $n > 0$ . Then  $\eta^n(A) \cong A$ . Moreover, any  $R$ -algebra  $\eta^n(A) \subseteq B \subseteq A$  is isomorphic to  $A$ , and there exist precisely  $2^n$  such intermediate  $R$ -algebras  $B$ .

Next we want to consider indecomposable rings  $R$  of arbitrary size. This will need the following definition.

**Definition 3.16.** *Let*

$$\mu_R = \begin{cases} \aleph_0, & \text{if } R \text{ is finite,} \\ \text{the smallest cardinal with a nontrivial} \\ |R|^+ \text{-additive } 0, 1\text{-valued measure,} & \text{if } R \text{ is infinite.} \end{cases}$$

For  $R$  infinite, we will set  $\mu_R = \infty$  in case that such a cardinal does not exist.

**Remark 3.17.**

- (1) For infinite  $R$ , we will briefly discuss the cardinal condition on  $\mu_R$ . First, note that  $\mu_R > |R|$  by definition, and that  $\mu_R$  must allow a nontrivial  $\sigma$ -additive  $0, 1$ -valued measure. Let  $\mu_0$  denote the smallest such cardinal. It is well-known that this cardinal  $\mu_0$  is measurable and inaccessible, cf. [6, Chapter 10]. Thus, the existence of  $\mu_R$  implies the existence of measurable cardinals in the chosen model of set theory, and it is consistent with ZFC that no measurable cardinals may exist. In this case  $\mu_R = \infty$  will apply, and statements with respect to cardinals  $\geq \mu_R$  will become void.
- (2) Note, that  $\mu : \mathcal{P}(P) \rightarrow \{0, 1\}$  is a nontrivial  $|R|^+$ -additive measure on a set  $P$  if and only if

$$\mathcal{U} = \{X \subseteq P : \mu(X) = 1\}$$

is a non-principal  $|R|^+$ -complete ultrafilter on  $P$ , cf. [6, Chapter 10]. Thus, in the following, all measure arguments are synonymous to ultrafilter arguments.

We note another immediate consequence of  $\mu_R$ .

**Proposition 3.18.** *For any sets  $R$  and  $P$  the following are equivalent:*

- (1) *There exists a nontrivial  $|R|^+$ -additive  $0, 1$ -valued measure on  $P$ .*
- (2)  $|P| \geq \mu_R$ .

*Proof.* For  $R$  finite, it is well-known that  $P$  allows a non-principal ultrafilter if and only if  $|P| \geq \aleph_0$ . For  $R$  infinite, (1) evidently implies (2).

Let now  $R$  be infinite and  $|P| \geq \mu_R$ . Choose a subset  $P' \subseteq P$  of cardinality  $|P'| = \mu_R$  and a nontrivial  $|R|^+$ -additive measure  $\mu' : \mathcal{P}(P') \rightarrow \{0, 1\}$  on  $P'$ . Then  $\mu(X) = \mu'(P' \cap X)$  for  $X \subseteq P$  defines a nontrivial  $|R|^+$ -additive  $0, 1$ -valued measure  $\mu$  on  $P$ .  $\square$

We are now ready for the main result of this section.

**Theorem 3.19.** *For any indecomposable ring  $R$  and  $\Pi = \prod_{x \in P} Re_x$  the following are equivalent:*

- (1)  $\text{LAut}(\Pi) = \text{Aut}(\Pi)$ .
- (2)  $|P| < \mu_R$ .

This theorem will be the collective achievement of a series of intermediate results. First, we would like to show that the existence of a nontrivial local automorphism implies  $|P| \geq \mu_R$ . The following auxiliary result will be crucial.

**Lemma 3.20.** *Let  $R$  be an infinite indecomposable ring and  $\Pi = \prod_{x \in P} Re_x$ . Let  $\gamma : \Pi \rightarrow R$  be some  $R$ -linear map such that*

$$(3.6) \quad \gamma(a) \in \{a_x : x \in P\} \text{ for all } a = \sum_{x \in P} a_x e_x \in \Pi \text{ and } \gamma(e_x) = 0 \text{ for all } x \in P.$$

*Then  $|P| \geq \mu_R$ .*

*Proof.* For  $X \subseteq P$  define  $e_X = \sum_{x \in X} e_x$ . Then  $e_\emptyset = 0$ , and  $e = e_P$  is the identity element of  $\Pi$ . Note that by (3.6) we have  $\gamma(e_X) \in \{0, 1\}$  for all  $X \subseteq P$ . We will identify  $0, 1 \in R$  with their real-valued counterparts, and define  $\mu : \mathcal{P}(P) \rightarrow \{0, 1\}$  by  $\mu(X) = \gamma(e_X)$  for all  $X \subseteq P$ .

To prove  $|P| \geq \mu_R$ , we will show that  $\mu$  is a nontrivial  $|R|^+$ -additive measure on  $P$ :

Obviously, by (3.6),

$$(3.7) \quad \mu(\emptyset) = \gamma(0) = 0, \mu(P) = \gamma(e) = 1, \text{ and } \mu(\{x\}) = \gamma(e_x) = 0 \text{ for all } x \in P.$$

Let  $X \subseteq P$ . Then

$$1 = \gamma(e) = \gamma(e_X + e_{P \setminus X}) = \gamma(e_X) + \gamma(e_{P \setminus X}) = \mu(X) + \mu(P \setminus X)$$

Thus

$$(3.8) \quad \mu(P \setminus X) = 1 - \mu(X) \text{ for all } X \subseteq P.$$

We claim that

$$(3.9) \quad \mu(X) = \mu(Y) = 1 \text{ for } X, Y \subseteq P \text{ implies } X \cap Y \neq \emptyset.$$

Assume  $\mu(X) = \mu(Y) = 1$  and  $X \cap Y = \emptyset$ . Choose  $f \in R \setminus \{0\}$  and  $g \in R \setminus \{0, -f\}$ . Then

$$f + g = f\mu(X) + g\mu(Y) = f\gamma(e_X) + g\gamma(e_Y) = \gamma(fe_X + ge_Y) \in \{0, f, g\}$$

by (3.6), and thus  $f + g \in \{0, f, g\}$ , a contradiction to our choice of  $f$  and  $g$ .

We next claim that

$$(3.10) \quad \mu(X) \leq \mu(Y) \text{ for all } X \subseteq Y \subseteq P.$$

Assume  $\mu(X) > \mu(Y)$ , thus  $\mu(X) = 1$  and  $\mu(Y) = 0$ . With (3.8) we have  $\mu(P \setminus Y) = 1 = \mu(X)$  with  $(P \setminus Y) \cap X = \emptyset$ , a contradiction to (3.9).

So far, we have shown that  $\mu$  is a nontrivial measure on  $P$ . It remains to show that  $\mu$  is  $|R|^+$ -additive:

To this end, let  $\kappa \leq |R|$  be a cardinal and  $X_\alpha \subseteq P$  for  $\alpha < \kappa$  be mutually disjoint sets with  $X = \bigcup_{\alpha < \kappa} X_\alpha$ , and assume that  $\mu(X) = 1$  and  $\mu(X_\alpha) = 0$  for all  $\alpha < \kappa$ . Set  $X_\kappa = P \setminus X$ , and observe that  $\mu(X_\kappa) = 0$  by (3.8). Thus, we have a partition

$$(3.11) \quad P = \bigcup_{\alpha \leq \kappa} X_\alpha \text{ with } \mu(X_\alpha) = 0 \text{ for all } \alpha \leq \kappa.$$

Pick  $a_\alpha \in R$  such that  $0 \neq a_\alpha \neq a_\beta$  for all  $\alpha \neq \beta \leq \kappa$  and let  $a = \sum_{\alpha \leq \kappa} a_\alpha e_{X_\alpha}$ . By (3.6) there is a unique  $\beta$  such that  $\gamma(a) = a_\beta$ . Now define

$$b = a - a_\beta e_{X_\beta} = \sum_{\substack{\alpha \leq \kappa \\ \alpha \neq \beta}} a_\alpha e_{X_\alpha}.$$

Then  $\gamma(b) \in \{0, a_\alpha : \beta \neq \alpha \leq \kappa\}$  with (3.6), but

$$\gamma(b) = \gamma(a - a_\beta e_{X_\beta}) = \gamma(a) - a_\beta \gamma(e_{X_\beta}) = \gamma(a) - a_\beta \mu(X_\beta) = \gamma(a) = a_\beta$$

with (3.11), a contradiction to our choice of  $a_\beta$ .  $\square$

We are all set for proving that Property (2) implies Property (1) in Theorem 3.19.

**Lemma 3.21.** *Let  $R$  be an indecomposable ring such that  $\text{LAut}(\Pi) \neq \text{Aut}(\Pi)$  for  $\Pi = \prod_{x \in P} Re_x$ . Then  $|P| \geq \mu_R$ .*

*Proof.* The inclusion  $\text{Aut}(\Pi) \subseteq \text{LAut}(\Pi)$  is obvious. Thus  $\text{LAut}(\Pi) \neq \text{Aut}(\Pi)$  implies the existence of some  $\eta \in \text{LAut}(\Pi) \setminus \text{Aut}(\Pi)$ .

Let  $E = \{e_x : x \in P\}$ . Then  $E$  is the set of all primitive idempotents of the  $R$ -algebra  $\Pi$  and thus  $\eta(E) \subseteq E$ . We infer that there exists a map  $\rho : P \rightarrow P$  such that  $\eta(e_x) = e_{\rho(x)}$  for all  $x \in P$ . Since  $\eta$  is injective, the map  $\rho$  is injective.

Suppose, for the moment, that  $P$  is finite. Then for all  $a = \sum_{x \in P} a_x e_x \in \Pi$  holds

$$\eta(a) = \eta\left(\sum_{x \in P} a_x e_x\right) = \sum_{x \in P} \eta(a_x e_x) = \sum_{x \in P} a_x \eta(e_x) = \sum_{x \in P} a_x e_{\rho(x)} = \widehat{\rho}(a),$$

thus  $\eta = \widehat{\rho}$ . Furthermore,  $E$  is finite, and  $\eta(E) \subseteq E$  implies  $\eta(E) = E$ . Thus  $\rho$  is a permutation of  $P$ , and  $\eta = \widehat{\rho} \in \text{Aut}(\Pi)$  with Proposition 3.2, a contradiction.

Therefore,  $P$  is infinite. For  $R$  finite, this implies  $|P| \geq \aleph_0 = \mu_R$  and the proof is complete. Hence, let  $R$  be infinite.

Suppose, for the moment, that  $\rho$  is surjective. Then  $\eta = \widehat{\rho} \in \text{Aut}(\Pi)$  follows as in Theorem 3.6, a contradiction.

Therefore,  $\rho$  is non-surjective, and we may pick some  $y \in P \setminus \text{Im}(\rho)$ . Let  $\pi : \Pi \rightarrow Re_y \cong R$  be the natural projection onto the  $e_y$ -coordinate. Let  $\gamma = \pi \circ \eta$ . Since  $\eta$  is a local automorphism, there exists for each  $a = \sum_{x \in P} a_x e_x \in \Pi$  some permutation  $\rho_a$  such that  $\eta(a) = \widehat{\rho}_a(a) = \sum_{x \in P} a_x e_{\rho_a(x)}$ . Thus

$$(3.12) \quad \gamma(a) = a_{\rho_a^{-1}(y)} \in \{a_x : x \in P\}.$$

Furthermore, for all  $x \in P$

$$(3.13) \quad \gamma(e_x) = \pi(\eta(e_x)) = \pi(e_{\rho(x)}) = 0$$

since  $y \notin \text{Im}(\rho)$ . With (3.12) and (3.13), the  $R$ -linear map  $\gamma$  has Property (3.6), and Lemma 3.20 yields  $|P| \geq \mu_R$ .  $\square$

Next follows another generalization of the ultrafilter construction of Lemma 3.13.

**Theorem 3.22.** *Let  $R$  be an indecomposable ring and  $P$  be a set with  $|P| \geq \mu_R$ . Then for  $\Pi = \prod_{x \in P} Re_x$  there exists some unital  $\eta \in \text{End}(\Pi)$  such that  $\eta$  is an  $n$ -local automorphism for all  $n > 0$ , but not surjective. In particular,  $\eta$  is not an  $R$ -algebra automorphism of  $\Pi$ .*

*Proof.* The case of finite  $R$  is covered by Lemma 3.13 and Corollary 3.14. Thus let  $R$  be infinite. As  $|P| \geq \mu_R$ , we may choose a nontrivial  $|R|^+$ -additive measure  $\mu : \mathcal{P}(P) \rightarrow \{0, 1\}$ , cf. Proposition 3.18. Let  $a \in \Pi$ . Then there exist disjoint sets  $[a]_r \subseteq P$ ,  $r \in R$ , such that  $a = \sum_{r \in R} r e_{[a]_r}$ . Since  $\mu$  is  $|R|^+$ -additive, there exists exactly one  $s =: \varphi(a) \in R$  such that  $\mu([a]_s) = 1$ . It is easy to check that  $\varphi : \Pi \rightarrow R$  is an  $R$ -algebra homomorphism. With respect to some enumeration  $\{x_\alpha : \alpha < |P|\}$  of  $P$  define  $\eta : \Pi \rightarrow \Pi$  by

$$\eta(a) = \varphi(a) e_{x_0} + \sum_{\alpha < |P|} a_{x_\alpha} e_{x_{\alpha+1}}.$$

for all  $a = \sum_{x \in P} a_x e_x \in \Pi$ . As in Lemma 3.13 and Corollary 3.14, we can show that  $\eta$  has all the necessary properties. Note, in particular, that

$$[a]_{\varphi(a)} \in \mathcal{U} = \{X \subseteq P : \mu(X) = 1\},$$

where  $\mathcal{U}$  is a non-principal  $|R|^+$ -complete ultrafilter on  $P$ , cf. Remark 3.17(2).  $\square$

We mention the following obvious generalization of Corollary 3.14.

**Corollary 3.23.** *If  $R$  is an indecomposable ring and  $P$  a set with  $|P| \geq \mu_R$ , then there exists an injective unital  $R$ -algebra endomorphism  $\eta$  of  $\Pi = \prod_{x \in P} Re_x$  such that  $\eta \neq \widehat{\rho}$  for all injective maps  $\rho : P \rightarrow P$ .*

Theorem 3.19 now follows from Lemma 3.21 and Theorem 3.22.

**3.4.  $R$ -algebra endomorphisms.** In this section we will revisit and generalize the idea that maps  $\rho : P \rightarrow P$  induce  $R$ -algebra endomorphisms  $\hat{\rho} \in \text{End}(\Pi)$ . Our reward will be a characterization of  $R$ -algebra endomorphisms in terms of  $\mu_R$ , cf. Theorem 3.28.

We start with a simple definition. As motivation, consider an arbitrary  $R$ -algebra endomorphism  $\eta \in \text{End}(\Pi)$ . For each  $x \in P$ , the element  $\eta(e_x)$  is an idempotent, and there exists some subset  $A_x \subseteq P$  such that

$$\eta(e_x) = e_{A_x}.$$

Note, that this includes the possible options  $A_x = \emptyset$  for  $\eta(e_x) = 0$  and  $A_x = P$  for  $\eta(e_x) = e$ , the identity element of  $\Pi$ . Furthermore, for  $x, y \in P$  with  $x \neq y$  we have

$$(3.14) \quad e_{A_x \cap A_y} = e_{A_x} e_{A_y} = \eta(e_x) \eta(e_y) = \eta(e_x e_y) = \eta(0) = 0,$$

hence  $A_x \cap A_y = \emptyset$ . It follows that  $\{A_x : x \in P\}$  is a family of pairwise disjoint subsets of  $P$ .

**Definition 3.24.** Let  $\eta \in \text{End}(\Pi)$ . We call  $\eta$  induced, if there exists a family  $\mathcal{F} = \{A_x : x \in P\}$  of pairwise disjoint subsets of  $P$  such that

$$(3.15) \quad \eta(a) = \sum_{x \in P} a_x e_{A_x} \quad \text{for all } a = \sum_{x \in P} a_x e_x \in \Pi.$$

In this case, we will call  $\eta = \eta_{\mathcal{F}}$  induced by  $\mathcal{F}$ .

It is readily verified that Equation (3.15) indeed defines an  $R$ -algebra homomorphism for every family  $\mathcal{F} = \{A_x : x \in P\}$  of pairwise disjoint subsets of  $P$ . The induced homomorphism  $\eta_{\mathcal{F}}$  is injective if and only if  $A_x \neq \emptyset$  for all  $x \in P$ . Note also, that for every injective map  $\rho : P \rightarrow P$  the  $R$ -algebra homomorphism  $\hat{\rho}$  is induced by the family

$$\mathcal{F} = \{A_x : x \in P\} \quad \text{with } A_x = \{\rho(x)\}.$$

Finally, for every induced  $R$ -algebra homomorphism  $\eta$  we can uniquely recover the inducing family  $\mathcal{F} = \{A_x : x \in P\}$  from Equation (3.14).

The following propositions list some more results on induced homomorphisms.

**Proposition 3.25.** Let  $\eta \in \text{End}(\Pi)$  with  $\eta(e_x) = e_{A_x}$  for all  $x \in P$ . If  $P = \bigcup_{x \in P} A_x$ , then  $\eta$  is induced by  $\mathcal{F} = \{A_x : x \in P\}$ .

*Proof.* Let  $a = \sum_{x \in P} a_x e_x \in \Pi$ . Then  $e = \sum_{x \in P} e_{A_x}$  and  $\eta(a) = \sum_{x \in P} \eta(a) e_{A_x}$ . Moreover,

$$\eta(a) e_{A_x} = \eta(a) \eta(e_x) = \eta(a e_x) = \eta(a_x e_x) = a_x \eta(e_x) = a_x e_{A_x},$$

and it follows that  $\eta(a) = \sum_{x \in P} a_x e_{A_x}$ . This shows that  $\eta$  is induced by  $\mathcal{F}$ .  $\square$

More generally still, the following holds.

**Proposition 3.26.** Let  $\eta \in \text{End}(\Pi)$  with  $\eta(e_x) = e_{A_x}$  for all  $x \in P$ , let  $A = \bigcup_{x \in P} A_x$ ,  $B = P \setminus A$  and  $\Pi = \Pi_A \oplus \Pi_B$  with  $\Pi_A = \prod_{x \in A} R e_x$  and  $\Pi_B = \prod_{x \in B} R e_x$ . Then for  $\eta = \eta_A \oplus \eta_B$  with  $\eta_A, \eta_B \in \text{End}(\Pi)$  the  $\Pi_A$ - and  $\Pi_B$ -component of  $\eta$ , respectively, the following holds.

- (a)  $\eta_A$  is induced by  $\mathcal{F} = \{A_x : x \in P\}$ .
- (b)  $\eta_B(e_x) = 0$  for all  $x \in P$ .
- (c)  $\eta$  is induced if and only if  $\eta_B = 0$ . In this case,  $\eta = \eta_A = \eta_{\mathcal{F}}$ .

We can strengthen the result of Corollary 3.23 as follows.

**Corollary 3.27.** *If  $R$  is an indecomposable ring and  $P$  a set with  $|P| \geq \mu_R$ , then there exists an injective unital  $R$ -algebra endomorphism  $\eta$  of  $\Pi = \prod_{x \in P} Re_x$  such that  $\eta$  is not induced.*

*Proof.* Let  $\{x_\alpha : \alpha < |P|\}$  be some enumeration of  $P$ . Let the map  $\varphi : \Pi \rightarrow R$  be defined via a non-principal  $|R|^+$ -complete ultrafilter on  $P$  as in Theorem 3.22, and define

$$\eta(a) = \varphi(a)e_{x_0} + \sum_{\alpha < |P|} a_{x_\alpha} e_{x_{\alpha+1}}$$

for all  $a = \sum_{x \in P} a_x e_x \in \Pi$ . Then  $\eta$  is an injective unital  $R$ -algebra endomorphism of  $\Pi$ . Moreover,  $\eta(e_{x_\alpha}) = e_{x_{\alpha+1}}$  and  $A_{x_\alpha} = \{x_{\alpha+1}\}$  for all  $\alpha < |P|$ . As  $x_0 \notin \bigcup_{x \in P} A_x$ , but  $\eta$  followed by the natural projection onto the  $e_{x_0}$ -coordinate is non-zero,  $\eta$  is not induced.  $\square$

The central result of this section will be the following adaption of Theorem 3.19 to the situation of  $R$ -algebra endomorphisms of  $\Pi$ . We will require  $R$  to be a field.

**Theorem 3.28.** *For any field  $R$  and  $\Pi = \prod_{x \in P} Re_x$  the following are equivalent:*

- (1) *Every  $\eta \in \text{End}(\Pi)$  is induced.*
- (2)  *$|P| < \mu_R$ .*

*Proof.* Based on Corollary 3.27 we just need to check that (2) implies (1). Let us therefore assume that there exists some  $\eta \in \text{End}(\Pi)$  which is not induced.

Decompose  $\eta = \eta_A \oplus \eta_B$  as in Proposition 3.26. As  $\eta$  is not induced, we have  $\eta_B \neq 0$ . In particular, we may pick some  $y \in B = P \setminus A$  such that  $\gamma := \pi \circ \eta \neq 0$ , where  $\pi : \Pi \rightarrow Re_y \cong R$  is the natural projection onto the  $e_y$ -coordinate. Note that  $\gamma$  is an  $R$ -algebra homomorphism.

With Proposition 3.26(b),

$$(3.16) \quad \gamma(e_x) = \pi(\eta(e_x)) = \pi(\eta_B(e_x)) = 0 \quad \text{for all } x \in P.$$

We claim

$$(3.17) \quad \gamma(e) = 1.$$

Assume  $\gamma(e) \neq 1$ . In this case, we have  $\gamma(e) = 0$ , the only other idempotent in  $R$ . Hence,

$$\gamma(a) = \gamma(ea) = \gamma(e)\gamma(a) = 0$$

for all  $a \in \Pi$ , a contradiction to our choice of  $y$ .

We next claim

$$(3.18) \quad \gamma(a) \in \{a_x : x \in P\} \quad \text{for all } a = \sum_{x \in P} a_x e_x \in \Pi.$$

To see this, consider the element  $b = a - \gamma(a)e$ . We have

$$\gamma(b) = \gamma(a - \gamma(a)e) = \gamma(a) - \gamma(\gamma(a)e) = \gamma(a) - \gamma(a)\gamma(e) = \gamma(a) - \gamma(a) = 0.$$

If  $b \in \Pi$  were invertible, this would yield

$$1 = \gamma(e) = \gamma(b)\gamma(b^{-1}) = 0,$$

a contradiction. Thus  $b$  must not be invertible. Hence, as  $R$  is a field, one coordinate of  $b = a - \gamma(a)e$  must be 0, and thus  $\gamma(a) \in \{a_x : x \in P\}$ .

With (3.16) and (3.18), the  $R$ -linear map  $\gamma$  has Property (3.6), and Lemma 3.20 yields  $|P| \geq \mu_R$ .  $\square$



## 4. LOCAL AUTOMORPHISMS OF FINITARY INCIDENCE ALGEBRAS

In the following, we will investigate the local automorphisms of finitary incidence algebras. Let us first fix some basic notation for the remainder of this section.

**Notation 4.1.** *Let  $(P, \leq)$  be a poset, and let  $R$  be a commutative, indecomposable ring. Then  $FI(P)$  will denote the induced finitary incidence algebra. We will associate any element  $a \in FI(P)$  with its standard representation  $a = \sum_{x \leq y} a_{xy} e_{xy}$ . For convenience we may write  $e_x$  instead of  $e_{xx}$ , and  $a_x$  instead of  $a_{xx}$ .*

We will also need the following definition.

**Definition 4.2.** *Let*

$$Z = Z(FI(P)) = \{a : a_x = 0 \text{ for all } x \in P\} \subseteq FI(P).$$

We gather some of the basic properties of  $Z(FI(P))$  for later use.

**Proposition 4.3.** *For  $Z = Z(FI(P))$  the following holds.*

- (a)  $Z \triangleleft FI(P)$  is a two-sided ideal.
- (b)  $FI(P)/Z \cong \prod_{x \in P} Re_x$ .
- (c)  $\varphi(Z) = Z$  for all  $\varphi \in \text{Aut}(FI(P))$ .

*Proof.* Parts (a) and (b) are easy to check. For (c), we just need to check  $\varphi(Z) \subseteq Z$ , as  $Z \subseteq \varphi(Z)$  follows from  $\varphi^{-1}(Z) \subseteq Z$ . Note, however, that  $\varphi = \psi_f \circ M_\sigma \circ \hat{\rho}$  with Theorem 2.3, and that  $\psi_f(Z) \subseteq Z$ ,  $M_\sigma(Z) \subseteq Z$  and  $\hat{\rho}(Z) \subseteq Z$  is immediate.  $\square$

Our main result will be the following generalization of Theorem 3.19.

**Theorem 4.4.** *Let  $(P, \leq)$  be a poset and  $R$  an indecomposable ring.*

- (a) *For  $|P| < \mu_R$ , we have  $\text{LAut}(FI(P)) = \text{Aut}(FI(P))$ .*
- (b) *For  $|P| \geq \mu_R$ , non-surjective local automorphisms may exist.*

Basically the same arguments apply to provide a corresponding generalization of Theorem 3.6.

**Theorem 4.5.** *Let  $(P, \leq)$  be a poset,  $R$  an indecomposable ring with  $|R| \geq 3$ , and  $\eta$  a local automorphism of  $FI(P)$  such that*

$$e_x \in \text{Im}(\eta) + Z(FI(P)) \quad \text{for all } x \in P.$$

*Then  $\eta$  is an  $R$ -algebra automorphism.*

This includes as an important special case the following result.

**Corollary 4.6.** *Let  $(P, \leq)$  be a poset and  $R$  an indecomposable ring with  $|R| \geq 3$ . Then every surjective local automorphism of  $FI(P)$  is an  $R$ -algebra automorphism.*

For part (b) of Theorem 4.4, we simply refer to Theorem 3.22 for a counterexample. Thus we only need to prove part (a), which will be the ultimate goal of a very elaborate chain of intermediate results. As a general agenda, we will try to mimic the proof of Theorem 2.3. Hence, confronted with an arbitrary  $\eta \in \text{LAut}(FI(P))$  we will attempt to split off suitable canonical automorphisms  $\hat{\rho}$ ,  $\psi_f$ , and  $M_\sigma$ , showing that the remaining local automorphism is the identity map on  $FI(P)$ .

#### 4.1. Step 1: Splitting off $\widehat{\rho}$ .

We will use Theorem 3.19 to isolate a hopeful candidate  $\rho : P \rightarrow P$  for a suitable order automorphism of  $(P, \leq)$ . Here, the cardinal condition  $|P| < \mu_R$  will become crucial in ascertaining that  $\rho$  is surjective.

**Proposition 4.7.** *Let  $|P| < \mu_R$  and  $\eta \in \text{LAut}(FI(P))$ . Then there exists a permutation  $\rho : P \rightarrow P$  such that*

$$(4.1) \quad \eta(a) \in \sum_{x \in P} a_x e_{\rho(x)} + Z(FI(P)) \quad \text{for all } a \in FI(P).$$

*Proof.* For any  $R$ -algebra automorphisms  $\varphi \in \text{Aut}(FI(P))$ , we have  $\varphi(Z) = Z$ , and  $\varphi$  induces a canonical  $R$ -algebra automorphism  $\overline{\varphi}$  on  $FI(P)/Z \cong \prod_{x \in P} Re_x$ . As a consequence, we have  $\eta(Z) \subseteq Z$ , and

$$\overline{\eta}(a + Z) = \eta(a) + Z \quad \text{for all } a \in FI(P)$$

induces a canonical local automorphism  $\overline{\eta}$  on  $FI(P)/Z$ . Applying Theorem 3.19 yields  $\overline{\eta} \in \text{Aut}(FI(P)/Z)$ , and with Proposition 3.2 there exists a permutation  $\rho : P \rightarrow P$  with

$$(4.2) \quad \overline{\eta} \left( \sum_{x \in P} a_x e_x + Z \right) = \sum_{x \in P} a_x e_{\rho(x)} + Z \quad \text{for all } a \in FI(P).$$

Equation (4.1) is now immediate.  $\square$

Replacing Theorem 3.19 in the last proof by Theorem 3.6 leads to the following corollary as a starting point for the proof of Theorem 4.5.

**Corollary 4.8.** *Let  $|R| \geq 3$ , and let  $\eta \in \text{LAut}(FI(P))$  with  $e_x \in \text{Im}(\eta) + Z(FI(P))$  for all  $x \in P$ . Then there exists a permutation  $\rho : P \rightarrow P$  such that*

$$\eta(a) \in \sum_{x \in P} a_x e_{\rho(x)} + Z(FI(P)) \quad \text{for all } a \in FI(P).$$

We need to show that the permutation  $\rho : P \rightarrow P$  in Proposition 4.7 is actually an order automorphism of  $(P, \leq)$ . This will need a more detailed knowledge of the structure of  $\text{Aut}(FI(P))$ . Our arguments will transfer immediately to Corollary 4.8.

**Lemma 4.9.** *Let  $|P| < \mu_R$ ,  $|R| \geq 3$ , and  $\eta \in \text{LAut}(FI(P))$ . Then there exists an order automorphism  $\rho$  of  $(P, \leq)$  such that  $\eta(a) \in \widehat{\rho}(a) + Z(FI(P))$  for all  $a \in FI(P)$ .*

*Proof.* We continue investigating the permutation  $\rho : P \rightarrow P$  from the proof of Proposition 4.7.

For any  $a \in FI(P)$ , there exists some  $\varphi_a \in \text{Aut}(FI(P))$  with  $\eta(a) = \varphi_a(a)$ . With Theorem 2.3 we have  $\varphi_a = \psi_{f_a} \circ M_{\sigma_a} \circ \widehat{\rho}_a$ , and

$$\overline{\eta}(a + Z) = \overline{\varphi_a}(a + Z) = \left( \overline{\psi_{f_a}} \circ \overline{M_{\sigma_a}} \circ \overline{\widehat{\rho}_a} \right) (a + Z) = \left( \overline{\psi_{f_a}} \circ \overline{M_{\sigma_a}} \right) \left( \sum_{x \in P} a_x e_{\rho_a(x)} + Z \right)$$

holds for the induced maps on  $FI(P)/Z$ . Note, however, that  $\psi_{f_a}$  and  $M_{\sigma_a}$  induce the identity map on  $FI(P)/Z$ . Thus,

$$(4.3) \quad \overline{\eta} \left( \sum_{x \in P} a_x e_x + Z \right) = \sum_{x \in P} a_x e_{\rho_a(x)} + Z \quad \text{for all } a \in FI(P).$$

Comparing (4.2) and (4.3) yields

$$(4.4) \quad \sum_{x \in P} a_x e_{\rho(x)} = \sum_{x \in P} a_x e_{\rho_a(x)} \quad \text{for all } a \in FI(P).$$

Let now  $x, y \in P$  be arbitrary distinct elements, and choose  $a_x, a_y \in R \setminus \{0\}$  with  $a_x \neq a_y$ . Application of (4.4) to the element  $a = a_x e_x + a_y e_y$  yields

$$a_x e_{\rho(x)} + a_y e_{\rho(y)} = a_x e_{\rho_a(x)} + a_y e_{\rho_a(y)}.$$

Thus, comparing coordinates, we have

$$(4.5) \quad \rho(x) = \rho_a(x) \quad \text{and} \quad \rho(y) = \rho_a(y).$$

Now, if  $x < y$ , then the order automorphism  $\rho_a$  of  $(P, \leq)$  yields  $\rho_a(x) < \rho_a(y)$ , and

$$\rho(x) = \rho_a(x) < \rho_a(y) = \rho(y).$$

Similarly,  $x > y$  yields  $\rho(x) > \rho(y)$ , while  $x, y$  incomparable yields  $\rho(x), \rho(y)$  incomparable, and  $\rho$  is an order automorphism.  $\square$

Once again, the situation  $|R| = 2$  has to be treated as an exceptional case and will need some new ideas. Note that for  $|R| = 2$  we have  $|P| < \mu_R = \aleph_0$ , and  $(P, \leq)$  is a finite poset. For any  $x \in P$ , let  $h(x)$  denote the *height of  $x$* , the size of a largest chain in  $(P, \leq)$  with maximal element  $x$ . Thus,  $h(x) = 1$  if and only if  $x$  is a minimal element of  $(P, \leq)$ . Note, that order automorphisms preserve heights.

The following lemma holds for indecomposable rings  $R$  of arbitrary size.

**Lemma 4.10.** *Let  $P$  be finite, and  $\eta \in \text{LAut}(FI(P))$ . Then there exists an order automorphism  $\rho$  of  $(P, \leq)$  such that  $\eta(a) \in \widehat{\rho}(a) + Z(FI(P))$  for all  $a \in FI(P)$ .*

*Proof.* We will make a more careful use of Equation (4.4).

First, consider the element  $a = e_x$  for some arbitrary element  $x \in P$ . With (4.4) we have  $e_{\rho(x)} = e_{\rho_a(x)}$ . Thus  $\rho(x) = \rho_a(x)$ , and as  $\rho_a$  preserves heights,

$$(4.6) \quad h(\rho(x)) = h(\rho_a(x)) = h(x) \quad \text{for all } x \in P.$$

Next, consider the element  $a = e_x + e_y$  for distinct elements  $x, y \in P$ . With (4.4) we have  $e_{\rho(x)} + e_{\rho(y)} = e_{\rho_a(x)} + e_{\rho_a(y)}$ . Thus, either

$$(4.7) \quad \rho(x) = \rho_a(x), \rho(y) = \rho_a(y) \quad \text{or} \quad \rho(x) = \rho_a(y), \rho(y) = \rho_a(x).$$

If  $x < y$ , then  $h(x) < h(y)$ , and the order automorphism  $\rho_a$  of  $(P, \leq)$  yields

$$h(\rho_a(x)) = h(x) < h(y) = h(\rho_a(y)).$$

However,  $\rho(x) = \rho_a(y), \rho(y) = \rho_a(x)$  yields with (4.4) that

$$h(\rho_a(x)) = h(\rho(y)) = h(y) > h(x) = h(\rho(x)) = h(\rho_a(y)),$$

a contradiction. Thus  $\rho(x) = \rho_a(x), \rho(y) = \rho_a(y)$  holds, and

$$\rho(x) = \rho_a(x) < \rho_a(y) = \rho(y).$$

Similarly,  $x > y$  yields  $\rho(x) > \rho(y)$ , while  $x, y$  incomparable yields  $\rho(x), \rho(y)$  incomparable, and  $\rho$  is an order automorphism.  $\square$

We can combine Lemmas 4.9 and 4.10 into one statement.

**Theorem 4.11.** *Let  $|P| < \mu_R$  and  $\eta \in \text{LAut}(FI(P))$ . Then there exists an order automorphism  $\rho$  of  $(P, \leq)$  such that  $\eta(a) \in \widehat{\rho}(a) + Z(FI(P))$  for all  $a \in FI(P)$ .*

In particular, replacing  $\eta$  with  $\widehat{\rho}^{-1} \circ \eta$ , we may, without loss of generality, assume that the local automorphism  $\eta$  induces the identity map on  $FI(P)/Z$ .

#### 4.2. Step 2: Splitting off $\psi_f$ .

For  $X \subseteq P$  define  $e_X = \sum_{x \in X} e_x$ . We start with some general observation on the structure of idempotents.

**Lemma 4.12.** *Let  $X \subseteq P$  and let*

$$a = \sum_{y \leq z} a_{yz} e_{yz} = e_X + \sum_{y < z} a_{yz} e_{yz} \in e_X + Z(FI(P))$$

*be an idempotent element. Then  $a_{yz} \neq 0$  implies  $y \leq x \leq z$  for some  $x \in X$ .*

*Proof.* Let  $u < v$  with  $a_{uv} \neq 0$  and consider the equation

$$(4.8) \quad a = a^n = \left( \sum_{x \in X} e_x + \sum_{y < z} a_{yz} e_{yz} \right)^n,$$

whose right-hand side must produce a nonzero coefficient at  $e_{uv}$ . Pick

$$(4.9) \quad n > |\{(y, z) : u \leq y < z \leq v, a_{yz} \neq 0\}|.$$

There exist elements  $w(0), w(1), \dots, w(n) \in P$  such that the product

$$e_{uv} = e_{w(0)w(1)} e_{w(1)w(2)} e_{w(2)w(3)} \cdots e_{w(n-1)w(n)}$$

makes a nonzero contribution

$$\prod_{i=0}^{n-1} a_{w(i)w(i+1)}$$

at  $e_{uv}$  to the right-hand side of (4.8). It follows that  $u = w(0)$ ,  $v = w(n)$  and  $w(i) \leq w(i+1)$  for all  $0 \leq i < n$ . If

$$x \notin \{u = w(0), w(1), \dots, w(n) = v\} \text{ for all } x \in X,$$

then  $w(i) < w(i+1)$  with  $a_{w(i)w(i+1)} \neq 0$  for all  $0 \leq i < n$ . Hence,

$$(w(i), w(i+1)) \in \{(y, z) : u \leq y < z \leq v, a_{yz} \neq 0\}$$

for all  $0 \leq i < n$ , and

$$|\{(y, z) : u \leq y < z \leq v, a_{yz} \neq 0\}| \geq n,$$

contradicting (4.9). It follows that  $x \in \{w(0), w(1), \dots, w(n)\}$  for some  $x \in X$ , and thus  $u = w(0) \leq x \leq w(n) = v$ .  $\square$

In the case of primitive idempotents we can be even more specific.

**Corollary 4.13.** *Fix  $x \in P$  and let*

$$a = \sum_{y \leq z} a_{yz} e_{yz} = e_x + \sum_{y < z} a_{yz} e_{yz} \in e_x + Z(FI(P))$$

*be an idempotent element. Then*

- (a)  $a_{uv} = \begin{cases} a_{ux} a_{xv}, & \text{for all } u \leq x \leq v, \\ 0, & \text{else.} \end{cases}$
- (b) *In particular,  $a = a e_x a$ .*

*Proof.* Note that

$$a = a^2 = \left( \sum_{u \leq v} a_{uv} e_{uv} \right)^2 = \sum_{u \leq z \leq v} a_{uz} a_{zv} e_{uv}.$$

Further note that  $a_{uz} a_{zv} \neq 0$  with Lemma 4.12 implies  $x \in [u, z] \cap [z, v] = \{z\}$ , and thus

$$a = \sum_{u \leq x \leq v} a_{ux} a_{xv} e_{uv}.$$

For part (b), simply observe that

$$a = \sum_{u \leq x \leq v} a_{ux} a_{xv} e_{ux} e_{xv} = \left( \sum_{u \leq x} a_{ux} e_{ux} \right) \left( \sum_{x \leq v} a_{xv} e_{xv} \right) = a e_x \cdot e_x a = a e_x a. \quad \square$$

We include the following result for a slightly different take on the same topic.

**Corollary 4.14.** *Fix  $x \in P$ . Then the following holds.*

- (a) *If  $a = b e_x c$  with  $b, c \in FI(P)$ , then  $a^2 = a_x a$ .*
- (b) *The element  $a \in e_x + Z(FI(P))$  is idempotent if and only if  $a = b e_x b$  for some  $b \in FI(P)$ .*

*Proof.* For (a), simply note  $a_x = b_x c_x$  and

$$a^2 = (b e_x c)(b e_x c) = b(e_x c b e_x)c = b(c_x b_x e_x)c = c_x b_x (b e_x c) = a_x a.$$

Part (b) is immediate from Corollary 4.13(b) and Corollary 4.14(a).  $\square$

We are all set to show the orthogonality of the primitive idempotents  $\eta(e_x)$ .

**Lemma 4.15.** *Let  $\eta \in \text{LAut}(FI(P))$  induce the identity map on  $FI(P)/Z$ . Then  $\eta(e_x)\eta(e_y) = 0$  holds for all  $x \neq y$ .*

*Proof.* We know that the idempotent elements  $\eta(e_x)$  and  $\eta(e_y)$  are of the forms

$$\eta(e_x) = \sum_{u \leq v} \alpha_{uv} e_{uv} = e_x + \sum_{u < v} \alpha_{uv} e_{uv} \text{ and } \eta(e_y) = \sum_{u \leq v} \beta_{uv} e_{uv} = e_y + \sum_{u < v} \beta_{uv} e_{uv}.$$

Note that

$$\eta(e_x)\eta(e_y) = \left( \sum_{u \leq v} \alpha_{uv} e_{uv} \right) \left( \sum_{u \leq v} \beta_{uv} e_{uv} \right) = \sum_{u \leq z \leq v} \alpha_{uz} \beta_{zv} e_{uv}.$$

We distinguish the following two cases.

**Case 1:**  $x \not\leq y$ .

Assume  $\eta(e_x)\eta(e_y) \neq 0$ . Pick  $u \leq v \in P$  such that  $(\eta(e_x)\eta(e_y))_{uv} \neq 0$ . Then there exists some  $u \leq z \leq v$  such that  $\alpha_{uz} \neq 0 \neq \beta_{zv}$ . By Lemma 4.12 we get  $u \leq x \leq z \leq y \leq v$  and thus  $x \leq y$ , a contradiction.

**Case 2:**  $x < y$ .

Since  $\eta$  preserves idempotents, considering the idempotents  $e_x$ ,  $e_y$  and  $e_x + e_y$ , we have

$$\begin{aligned} \eta(e_x) + \eta(e_y) &= \eta(e_x + e_y) = \eta(e_x + e_y)^2 = (\eta(e_x) + \eta(e_y))^2 \\ &= \eta(e_x)^2 + \eta(e_y)^2 + \eta(e_x)\eta(e_y) + \eta(e_y)\eta(e_x) \\ &= \eta(e_x) + \eta(e_y) + \eta(e_x)\eta(e_y) + \eta(e_y)\eta(e_x), \end{aligned}$$

and

$$\eta(e_x)\eta(e_y) + \eta(e_y)\eta(e_x) = 0$$

follows. By Case 1, we have  $\eta(e_y)\eta(e_x) = 0$ , and thus  $\eta(e_x)\eta(e_y) = 0$ .  $\square$

We want to strengthen Lemma 4.15 to include  $\eta(e_x)\eta(e_Y) = \eta(e_Y)\eta(e_x) = 0$  for suitably chosen subsets  $Y \subseteq P$ . This will need a little bit of technical preparation. As usual, a subset  $S$  of a poset  $(P, \leq)$  has the *ascending chain condition* (acc) if it contains no infinite strictly ascending chain. Similarly,  $S$  has the *descending chain condition* (dcc) if it contains no infinite strictly descending chain.

**Lemma 4.16.** *Let a poset  $(P, \leq)$  and an infinite set  $S \subseteq P$  be given. Then there exists a sequence  $(y_i)_{i \in \omega}$  of elements in  $S$  that is either strictly ascending, or strictly descending, or consisting of pairwise incomparable elements in  $(P, \leq)$ .*

*Proof.* Let  $S$  be an infinite subset of  $P$ . Assume that  $S$  contains no infinite strictly ascending or strictly descending chains. Then  $S$  has the acc and dcc, and so does any subset of  $S$ .

Let  $S_0 = S$ , and let  $\max(S_0)$  denote the set of elements maximal in  $S_0$ . With acc, we have  $\max(S_0) \neq \emptyset$ . Assume  $\max(S_0)$  is finite. Then there exists some  $m_0 \in \max(S_0)$  such that  $S_1 = \{x \in S_0 : x < m_0\}$  is infinite. Assume  $\max(S_1) \neq \emptyset$  is finite. Then there exists some  $m_1 \in \max(S_1)$  such that  $S_2 = \{x \in S_1 : x < m_1\}$  is infinite. Continue the process. If  $\max(S_i)$  is finite for all  $i$ , then we obtain an infinite sequence  $m_0 > m_1 > \dots > m_i > m_{i+1} > \dots$ , contradicting dcc. Thus we must have encountered some  $i$  such that  $\max(S_i)$  is infinite, and thus an infinite set of pairwise incomparable elements has been found.  $\square$

We can put the constructed sequence  $(y_i)_{i \in \omega}$  to some good use to generalize the argumentation of Lemma 4.15.

**Lemma 4.17.** *Let  $\eta \in \text{LAut}(FI(P))$  induce the identity map on  $FI(P)/Z$ . Then for every infinite set  $S \subseteq P$  there exists some infinite set  $Y \subseteq S$  such that*

$$\eta(e_x)\eta(e_{Y \setminus \{x\}}) = \eta(e_{Y \setminus \{x\}})\eta(e_x) = 0$$

for all  $x \in Y$ .

*Proof.* With Lemma 4.16, we can choose a sequence  $(y_i)_{i \in \omega}$  of elements in  $S$  that is either strictly ascending, or strictly descending, or consisting of pairwise incomparable elements in  $(P, \leq)$ . Let  $Y = \{y_i \mid i \in \omega\} \subseteq S$ . We distinguish the following three cases.

**Case 1:** the elements  $y_i$  are pairwise incomparable.

Assume  $\eta(e_x)\eta(e_{Y \setminus \{x\}}) \neq 0$  for some  $x \in Y$ . Pick  $u \leq v \in P$  with  $(\eta(e_x)\eta(e_{Y \setminus \{x\}}))_{uv} \neq 0$ . Then there exists some  $u \leq z \leq v$  such that  $(\eta(e_x))_{uz} \neq 0 \neq (\eta(e_{Y \setminus \{x\}}))_{zv}$ . By Lemma 4.12 we get  $u \leq x \leq z \leq y \leq v$  for some  $x \neq y \in Y$ . Thus  $x < y$ , contradicting  $x, y$  being incomparable. Similarly,  $\eta(e_{Y \setminus \{x\}})\eta(e_x) = 0$  follows.

**Case 2:** the sequence  $(y_i)_{i \in \omega}$  is strictly ascending.

Assume  $\eta(e_{Y \setminus \{x\}})\eta(e_x) \neq 0$  for some  $x = y_j \in Y$ . With  $Y' = \{y_i \mid i > j\}$  and Lemma 4.15 we have

$$\begin{aligned} 0 \neq \eta(e_{Y \setminus \{x\}})\eta(e_x) &= \eta\left(e_{Y'} + \sum_{i=0}^{j-1} e_{y_i}\right)\eta(e_{y_j}) \\ &= \eta(e_{Y'})\eta(e_{y_j}) + \sum_{i=0}^{j-1} \eta(e_{y_i})\eta(e_{y_j}) = \eta(e_{Y'})\eta(e_x). \end{aligned}$$

Pick  $u \leq v \in P$  with  $(\eta(e_{Y'})\eta(e_x))_{uv} \neq 0$ . Then there exists some  $u \leq z \leq v$  such that  $(\eta(e_{Y'}))_{uz} \neq 0 \neq (\eta(e_x))_{zv}$ . By Lemma 4.12 we get  $u \leq y \leq z \leq x \leq v$  for some  $x \neq y = y_k \in Y'$ . Thus  $y_k = y < x = y_j$  with  $k > j$ , contradicting  $(y_i)_{i \in \omega}$  strictly ascending. This shows  $\eta(e_{Y \setminus \{x\}})\eta(e_x) = 0$ .

Since  $\eta$  preserves idempotents, considering the idempotents  $e_x$ ,  $e_{Y \setminus \{x\}}$  and  $e_Y$ , we have

$$\begin{aligned} \eta(e_x) + \eta(e_{Y \setminus \{x\}}) &= \eta(e_x + e_{Y \setminus \{x\}}) = \eta(e_Y) = \eta(e_Y)^2 = \eta(e_x + e_{Y \setminus \{x\}})^2 \\ &= \eta(e_x)^2 + \eta(e_{Y \setminus \{x\}})^2 + \eta(e_x)\eta(e_{Y \setminus \{x\}}) + \eta(e_{Y \setminus \{x\}})\eta(e_x) \\ &= \eta(e_x) + \eta(e_{Y \setminus \{x\}}) + \eta(e_x)\eta(e_{Y \setminus \{x\}}) + \eta(e_{Y \setminus \{x\}})\eta(e_x), \end{aligned}$$

and

$$\eta(e_x)\eta(e_{Y \setminus \{x\}}) + \eta(e_{Y \setminus \{x\}})\eta(e_x) = 0$$

follows. Thus,  $\eta(e_{Y \setminus \{x\}})\eta(e_x) = 0$  implies  $\eta(e_x)\eta(e_{Y \setminus \{x\}}) = 0$ .

**Case 3:** the sequence  $(y_i)_{i \in \omega}$  is strictly descending.

This case is handled similar to Case 2. □

As an immediate consequence, we note the following.

**Theorem 4.18.** *Let  $\eta \in \text{LAut}(FI(P))$  induce the identity map on  $FI(P)/Z$ . Then for every infinite set  $S \subseteq P$  there exists some infinite set  $Y \subseteq S$  such that*

$$\eta(e_x)\eta(e_Y) = \eta(e_Y)\eta(e_x) = \eta(e_x)$$

for all  $x \in Y$ .

*Proof.* Choosing  $Y$  as in Lemma 4.17, we have

$$\begin{aligned} \eta(e_x)\eta(e_Y) &= \eta(e_x)\eta(e_x + e_{Y \setminus \{x\}}) = \eta(e_x)(\eta(e_x) + \eta(e_{Y \setminus \{x\}})) \\ &= \eta(e_x)^2 + \eta(e_x)\eta(e_{Y \setminus \{x\}}) = \eta(e_x)^2 = \eta(e_x), \end{aligned}$$

and  $\eta(e_Y)\eta(e_x) = \eta(e_x)$  follows similarly. □

We will next introduce an element  $\beta \in I(P)$ , in terms of  $\eta$ , crucial to splitting off an inner automorphism  $\psi_\beta$  of  $\eta$ . Of course, it is essential to prove that  $\beta \in FI(P)$ . This requires quite some work even if  $\eta$  is an automorphism, cf. [7]. We have to develop some new ideas to obtain the same result for the **local** automorphism  $\eta$ .

**Lemma 4.19.** *Let  $\eta \in \text{LAut}(FI(P))$  induce the identity map on  $FI(P)/Z$ . Then*

$$\beta = \sum_{x \in P} e_x \eta(e_x)$$

defines an element  $\beta \in FI(P)$ .

*Proof.* Evidently,  $\beta_{xy} = (\eta(e_x))_{xy}$  for all  $x < y \in P$  by definition, and  $\beta$  is a well-defined element of the incidence space  $I(P)$ . We need to show  $\beta \in FI(P)$ .

Assume  $\beta \notin FI(P)$ . Then there exist  $a < b \in P$  for which the set

$$W = \{(u, v) : a \leq u < v \leq b, \beta_{uv} \neq 0\}$$

is infinite. Let

$$U = \{u \in P : \exists v \in P \text{ with } (u, v) \in W\},$$

and for all  $u \in U$  let

$$S_u = \{v \in P : (u, v) \in W\} \neq \emptyset.$$



If  $S_u$  is infinite for some  $u \in U$ , then  $\beta_{uv} = (\eta(e_u))_{uv} \neq 0$  for all  $a \leq v \leq b$  with  $v \in S_u$ , a contradiction to  $\eta(e_u) \in FI(P)$ . Thus,  $S_u \neq \emptyset$  is a finite set for all  $u \in U$ , and

$$T_u = \{v \in S_u : v \text{ is minimal in } S_u\} \neq \emptyset.$$

For  $v \in T_u$  holds  $\beta_{uv} = (\eta(e_u))_{uv} \neq 0$  but  $(\eta(e_u))_{ut} = 0$  for all  $u < t < v$ .

If the set  $U$  is finite, then there exists some  $u \in U$  with  $S_u$  infinite, contradiction. Thus  $U$  is infinite, and we will distinguish between the following two cases.

**Case 1:**  $I = \{u \in U : y \in T_u\} = \{u \in U : (\eta(e_u))_{uy} \neq 0\}$  is infinite for some  $y \in \bigcup_{u \in U} T_u$ .

We have  $y \notin I$  by definition, and with Theorem 4.18 we may assume

$$\eta(e_x) = \eta(e_x)\eta(e_I)$$

for all  $x \in I$ , replacing  $I$  by a suitable infinite subset of  $I$  if need be. We compute and compare the  $e_{xy}$ -coordinates of the terms in the last equation:

$$\begin{aligned} 0 &\neq (\eta(e_x))_{xy} = \sum_{x \leq t \leq y} (\eta(e_x))_{xt} (\eta(e_I))_{ty} \\ &= (\eta(e_x))_x (\eta(e_I))_{xy} + (\eta(e_x))_{xy} (\eta(e_I))_y + \sum_{x < t < y} (\eta(e_x))_{xt} (\eta(e_I))_{ty} \end{aligned}$$

As  $\eta$  induces the identity map on  $FI(P)/Z$ , we have  $(\eta(e_x))_x = 1$  and  $(\eta(e_I))_y = (e_I)_y = 0$  since  $y \notin I$ . Furthermore, as  $y \in T_x$ , we have  $(\eta(e_x))_{xt} = 0$  whenever  $x < t < y$ . We obtain

$$0 \neq (\eta(e_x))_{xy} = (\eta(e_I))_{xy}$$

for all  $a \leq x < y \leq b$  with  $x \in I$ , a contradiction to  $\eta(e_I) \in FI(P)$ .

**Case 2:**  $\{u \in U : v \in T_u\}$  is finite for all  $v \in \bigcup_{u \in U} T_u$ .

We construct recursively distinct elements  $x_i, y_i$  ( $i \in \omega$ ) with  $y_i \in T_{x_i}$ . Start with arbitrary elements  $x_0 \in U$  and  $y_0 \in T_{x_0}$ . Given elements  $x_j, y_j$  ( $j \leq i$ ), choose

$$x_{i+1} \in U \setminus \left( \{x_j, y_j : j \leq i\} \cup \bigcup_{j \leq i} \{u \in U : x_j \in T_u\} \cup \bigcup_{j \leq i} \{u \in U : y_j \in T_u\} \right)$$

and  $y_{i+1} \in T_{x_{i+1}}$ . We have  $(x_i, y_i) \in W$  by definition. Let  $J = \{x_i : i < \omega\}$ . With Theorem 4.18 we may assume

$$\eta(e_{x_i}) = \eta(e_{x_i})\eta(e_J)$$

for all  $i \in \omega$ , replacing  $J$  by a suitable infinite subset of  $J$  if need be. We compute and compare the  $e_{x_i y_i}$ -coordinates of the terms in the last equation:

$$\begin{aligned} 0 &\neq (\eta(e_{x_i}))_{x_i y_i} = \sum_{x_i \leq t \leq y_i} (\eta(e_{x_i}))_{x_i t} (\eta(e_J))_{ty_i} \\ &= (\eta(e_{x_i}))_{x_i} (\eta(e_J))_{x_i y_i} + (\eta(e_{x_i}))_{x_i y_i} (\eta(e_J))_{y_i} + \sum_{x_i < t < y_i} (\eta(e_{x_i}))_{x_i t} (\eta(e_J))_{ty_i} \end{aligned}$$

As  $\eta$  induces the identity map on  $FI(P)/Z$ , we have  $(\eta(e_{x_i}))_{x_i} = 1$  and  $(\eta(e_J))_{y_i} = (e_J)_{y_i} = 0$  since  $y_i \notin J$ . Furthermore, as  $y_i \in T_{x_i}$ , we have  $(\eta(e_{x_i}))_{x_i t} = 0$  whenever  $x_i < t < y_i$ . We obtain

$$0 \neq (\eta(e_{x_i}))_{x_i y_i} = (\eta(e_J))_{x_i y_i}$$

for infinitely many distinct pairs  $(x_i, y_i) \in W$  with  $a \leq x_i < y_i \leq b$ , a contradiction to  $\eta(e_J) \in FI(P)$ .  $\square$

We are all set for the main result of this section.

**Theorem 4.20.** *Let  $\eta \in \text{LAut}(FI(P))$  induce the identity map on  $FI(P)/Z$ . Then there exists a unit  $\beta \in FI(P)$  with  $\eta(e_x) = \psi_\beta(e_x)$  for all  $x \in P$ .*

*Proof.* With Lemma 4.19, we can choose  $\beta = \sum_{x \in P} e_x \eta(e_x) \in FI(P)$ . Note that

$$\beta_x = (\eta(e_x))_x = 1$$

is a unit of  $R$  for all  $x \in P$ , which makes  $\beta$  a unit of  $FI(P)$ , cf. Section 2. Furthermore, with Lemma 4.15 we have

$$\begin{aligned} \beta \eta(e_x) &= \sum_{y \in P} e_y \eta(e_y) \eta(e_x) = e_x \eta(e_x)^2 = e_x \eta(e_x) \\ &= e_x^2 \eta(e_x) = e_x \sum_{y \in P} e_y \eta(e_y) = e_x \beta \end{aligned}$$

for all  $x \in P$ . Hence,  $\eta(e_x) = \beta^{-1} e_x \beta = \psi_\beta(e_x)$ .  $\square$

In particular, replacing  $\eta$  with  $\psi_\beta^{-1} \circ \eta$ , we may, without loss of generality, assume that the local automorphism  $\eta$  induces the identity map on  $FI(P)/Z$  with  $\eta(e_x) = e_x$  for all  $x \in P$ .

### 4.3. Step 3: Splitting off $M_\sigma$ .

Splitting off a suitable Schur multiplication  $M_\sigma$  will be a refreshingly simple task.

**Theorem 4.21.** *Let  $\eta \in \text{LAut}(FI(P))$  with  $\eta(e_x) = e_x$  for all  $x \in P$ . Then the following holds.*

- (a) *For all  $x \leq y \in P$ ,  $\eta(e_{xy}) = \sigma_{xy} e_{xy}$  with  $\sigma_{xy} \in R$ .*
- (b) *For all  $x \leq y \in P$ ,  $\sigma_{xy}$  is a unit of  $R$ . Moreover,  $\sigma_{xx} = 1$  for all  $x \in P$ .*
- (c) *For all  $x \leq y \leq z \in P$ ,  $\sigma_{xz} = \sigma_{xy} \sigma_{yz}$ .*

In particular,  $\sigma$  induces a Schur multiplication  $M_\sigma$  with  $\eta(e_{xy}) = M_\sigma(e_{xy})$  for all  $x \leq y \in P$ .

*Proof.* For (a), we have  $\eta(e_{xx}) = e_{xx} \in Re_{xx}$ . Thus, we may assume  $x < y$ .

Applying  $\eta$  to the idempotent  $e_x + e_{xy}$  gives

$$\begin{aligned} e_x + \eta(e_{xy}) &= \eta(e_x) + \eta(e_{xy}) = \eta(e_x + e_{xy}) = \eta(e_x + e_{xy})^2 = (e_x + \eta(e_{xy}))^2 \\ &= e_x + \eta(e_{xy})^2 + e_x \eta(e_{xy}) + \eta(e_{xy}) e_x. \end{aligned}$$

Moreover, choosing some  $\varphi \in \text{Aut}(FI(P))$  with  $\eta(e_{xy}) = \varphi(e_{xy})$ , we have  $\eta(e_{xy})^2 = \varphi(e_{xy})^2 = \varphi(e_{xy}^2) = \varphi(0) = 0$ , and the last equation simplifies to

$$(4.10) \quad \eta(e_{xy}) = e_x \eta(e_{xy}) + \eta(e_{xy}) e_x.$$

Similarly, from the idempotent  $e_y + e_{xy}$  we infer

$$(4.11) \quad \eta(e_{xy}) = e_y \eta(e_{xy}) + \eta(e_{xy}) e_y.$$

Comparing coordinates on both sides of Equation (4.10), we have  $(\eta(e_{xy}))_{uv} = 0$  for  $u \leq v \in P$  unless either  $u = x$  or  $v = x$ . Similarly, (4.11) gives  $(\eta(e_{xy}))_{uv} = 0$  for  $u \leq v \in P$  unless either  $u = y$  or  $v = y$ . Thus,  $(\eta(e_{xy}))_{xy}$  is the only possible nontrivial entry of  $\eta(e_{xy})$ , and  $\eta(e_{xy}) \in Re_{xy}$ .

For (b), let  $\eta(e_{xy}) = \sigma_{xy} e_{xy}$  and choose some  $\varphi \in \text{Aut}(FI(P))$  with  $\eta(e_{xy}) = \varphi(e_{xy})$ . With  $a = \varphi^{-1}(e_{xy}) \in FI(P)$  we have

$$\varphi(e_{xy}) = \eta(e_{xy}) = \sigma_{xy} e_{xy} = \sigma_{xy} \varphi(a) = \varphi(\sigma_{xy} a).$$

Hence  $e_{xy} = \sigma_{xy}a$ , and looking at the  $e_{xy}$ -coordinates of this equation gives  $1 = \sigma_{xy}a_{xy}$ . Thus,  $\sigma_{xy}$  is a unit, with  $\sigma_{xx} = 1$  evident from  $\eta(e_{xx}) = e_{xx}$ .

For (c), with  $\sigma_{yy} = 1$  the statement trivially holds if  $x = y$  or  $y = z$ , and we need to check  $\sigma_{xz} = \sigma_{xy}\sigma_{yz}$  only for the case  $x < y < z$ . For that purpose, we will investigate the idempotent element  $a = e_y + e_{xy} + e_{yz} + e_{xz}$ . First, note that

$$a = e_y + e_{xy} + e_{yz} + e_{xz} = (e_y + e_{xy})(e_y + e_{yz}) = ae_ya$$

confirms  $a$  as an idempotent, cf. Corollary 4.14. We have

$$\eta(a) = \eta(e_y) + \eta(e_{xy}) + \eta(e_{yz}) + \eta(e_{xz}) = e_y + \sigma_{xy}e_{xy} + \sigma_{yz}e_{yz} + \sigma_{xz}e_{xz},$$

and after squaring

$$\begin{aligned} \eta(a) = \eta(a)^2 &= (e_y + \sigma_{xy}e_{xy} + \sigma_{yz}e_{yz} + \sigma_{xz}e_{xz})^2 \\ &= e_y + \sigma_{xy}e_{xy} + \sigma_{yz}e_{yz} + \sigma_{xy}\sigma_{yz}e_{xz}. \end{aligned}$$

Comparing these last two equations, we infer  $\sigma_{xz} = \sigma_{xy}\sigma_{yz}$ .  $\square$

In particular, replacing  $\eta$  with  $M_\sigma^{-1} \circ \eta$ , we may, without loss of generality, assume that the local automorphism  $\eta$  induces the identity map on  $FI(P)/Z$  with  $\eta(e_{xy}) = e_{xy}$  for all  $x \leq y \in P$ . It remains to show  $\eta = \text{id}$ .

#### 4.4. Step 4: Finish.

It will be convenient to talk about diagonal elements.

**Definition 4.22.** We call  $d \in FI(P)$  diagonal if  $d_{xy} = 0$  for all  $x < y \in P$ .

The following lemma provides a first very powerful boost towards proving  $\eta = \text{id}$ .

**Lemma 4.23.** Let  $\eta \in \text{LAut}(FI(P))$  induce the identity map on  $FI(P)/Z$  with  $\eta(e_x) = e_x$  for all  $x \in P$ . Then  $\eta(d) = d$  for all diagonal elements  $d = \sum_{x \in P} d_x e_x$ .

*Proof.* We have  $\eta(d) = d + j$  for some  $j \in Z$ . For a contradiction, let us assume that  $j \neq 0$ . Thus, we can choose  $u < v \in P$  with  $j_{uv} \neq 0$ . The set

$$W = \{t : u < t \leq v, j_{ut} \neq 0\}$$

is finite. Without loss of generality, we may assume  $W = \{v\}$ , replacing  $v$  by  $\min W$  if need be. Thus,  $j_{uv} \neq 0$  but

$$(4.12) \quad j_{ut} = 0 \text{ for all } u < t < v.$$

Let  $E = P \setminus \{u, v\}$ , and set  $d^{(E)} = \sum_{x \in E} d_x e_x$ . We have

$$(4.13) \quad d = d_u e_u + d_v e_v + d^{(E)}.$$

Applying  $\eta$  gives

$$(4.14) \quad \eta(d) = \eta(d_u e_u + d_v e_v + d^{(E)}) = d_u e_u + d_v e_v + \eta(d^{(E)}).$$

On the other hand, we have

$$(4.15) \quad \eta(d) = d + j = d_u e_u + d_v e_v + d^{(E)} + j,$$

and we infer

$$(4.16) \quad \eta(d^{(E)}) = d^{(E)} + j$$

from comparing (4.14) and (4.15).

Choose some  $\varphi \in \text{Aut}(FI(P))$  with  $\eta(d^{(E)}) = \varphi(d^{(E)})$ . With Theorem 2.3 we have  $\varphi = \psi_f \circ M_\sigma \circ \hat{\rho}$ . Note that  $\hat{\rho}(d^{(E)})$  is a diagonal element and that diagonal elements are fixed under Schur multiplications. Hence

$$(4.17) \quad \eta(d^{(E)}) = (\psi_f \circ M_\sigma)(\hat{\rho}(d^{(E)})) = \psi_f(\hat{\rho}(d^{(E)})) = f^{-1}(\hat{\rho}(d^{(E)}))f,$$

and

$$\eta(d^{(E)}) + Z = f^{-1}(\hat{\rho}(d^{(E)}))f + Z = \hat{\rho}(d^{(E)}) + Z.$$

As  $\eta$  induce the identity map on  $FI(P)/Z$ , we infer  $\hat{\rho}(d^{(E)}) = d^{(E)}$ . Thus, (4.17) becomes

$$\eta(d^{(E)}) = f^{-1}d^{(E)}f$$

for some unit  $f \in FI(P)$ . Together with (4.16) we have

$$(4.18) \quad d^{(E)} + j = f^{-1}d^{(E)}f.$$

Now consider the  $e_{uv}$ -coordinate of the last equation. We have

$$0 \neq j_{uv} = (d^{(E)} + j)_{uv} = (f^{-1}d^{(E)}f)_{uv} = \sum_{u \leq t \leq v, t \in E} (f^{-1})_{ut} d_{tv}.$$

Note that in the latter summation we actually have  $u < t < v$  since  $u, v \notin E$ . This implies

$$(4.19) \quad (f^{-1})_{uw} d_{wv} f_{wv} \neq 0 \text{ for some } u < w < v.$$

With (4.18) we get

$$(4.20) \quad (d^{(E)} + j)(f^{-1}e_w f) = (f^{-1}d^{(E)}f)(f^{-1}e_w f) = f^{-1}d^{(E)}e_w f = f^{-1}d_w e_w f.$$

Considering the  $e_{uv}$ -coordinate of the last equation leads to

$$(d^{(E)}f^{-1}e_w f + jf^{-1}e_w f)_{uv} = (d_w f^{-1}e_w f)_{uv} = (f^{-1})_{uw} d_{wv} f_{wv} \neq 0.$$

Note that  $(d^{(E)}f^{-1}e_w f)_{uv} = 0$  since  $u \notin E$ , and we get

$$0 \neq (jf^{-1}e_w f)_{uv} = \sum_{u < t \leq w} j_{ut} (f^{-1})_{tw} f_{wv},$$

where we have a strict inequality  $u < t$  since  $j \in Z$ . It follows that  $j_{ut} \neq 0$  for some  $u < t \leq w < v$ , contradicting (4.12).  $\square$

For our final theorem we need one last crucial definition.

**Definition 4.24.** For any  $u \leq v \in P$  let

$$L_{uv} = \{a \in FI(P) : a_{xy} = 0 \text{ for all } u \leq x \leq y \leq v\}.$$

We summarize some of the remarkable properties of  $L_{uv}$  as a separate lemma.

**Lemma 4.25.** Let  $\eta \in \text{LAut}(FI(P))$  induce the identity map on  $FI(P)/Z$ . Then  $L_{uv} \triangleleft FI(P)$  is a two-sided ideal for all  $u \leq v \in P$ , and  $\eta(L_{uv}) \subseteq L_{uv}$  holds.

*Proof.* First we show that  $L_{uv}$  is a two-sided ideal. Let  $a \in L_{uv}$  and  $\gamma \in FI(P)$ . Then

$$(\gamma a)_{xy} = \sum_{x \leq t \leq y} \gamma_{xt} a_{ty}$$

for all  $x \leq y \in P$ . If  $u \leq x \leq y \leq v$ , then  $a_{ty} = 0$  for all  $x \leq t \leq y$ . Thus  $(\gamma a)_{xy} = 0$ , and we infer  $\gamma a \in L_{uv}$ . In a similar way,  $a\gamma \in L_{uv}$  follows.

For  $\eta(L_{uv}) \subseteq L_{uv}$ , start with some  $a \in L_{uv}$  such that

$$(4.21) \quad a_x = 0 \text{ if and only if } u \leq x \leq v.$$

Choose some  $\varphi \in \text{Aut}(FI(P))$  with  $\eta(a) = \varphi(a)$ . With Theorem 2.3 we have  $\varphi = \psi_f \circ M_\sigma \circ \hat{\rho}$ . By definition,  $\psi_f(L_{uv}) = f^{-1}L_{uv}f \subseteq L_{uv}$  and  $M_\sigma(L_{uv}) \subseteq L_{uv}$  are evident, and we only need to show that  $\hat{\rho}(a) \in L_{uv}$ . Note that  $\eta$  but also both  $\psi_f$  and  $M_\sigma$  induce the identity map on  $FI(P)/Z$ . Thus

$$\sum_{x \in P} a_x e_x + Z = a + Z = \eta(a) + Z = \hat{\rho}(a) + Z = \sum_{x \in P} a_x e_{\rho(x)} + Z = \sum_{x \in P} a_{\rho^{-1}(x)} e_x + Z,$$

and  $a_x = a_{\rho^{-1}(x)}$  follows for all  $x \in P$ . In particular, with (4.21) we have

$$u \leq x \leq v \iff a_x = 0 \iff a_{\rho^{-1}(x)} = 0 \iff u \leq \rho^{-1}(x) \leq v$$

and

$$u \leq x \leq y \leq v \implies u \leq \rho^{-1}(x) \leq \rho^{-1}(y) \leq v \implies a_{\rho^{-1}(x)\rho^{-1}(y)} = 0.$$

Thus

$$\hat{\rho}(a) = \sum_{x \leq y} a_{xy} e_{\rho(x)\rho(y)} = \sum_{x \leq y} a_{\rho^{-1}(x)\rho^{-1}(y)} e_{xy} \in L_{uv}.$$

This shows  $\hat{\rho}(a) \in L_{uv}$  and  $\eta(L_{uv}) \subseteq L_{uv}$  under condition (4.21).

Now let  $a$  be any element of  $L_{uv}$ . Note that

$$d = \left( \sum_{x \in P} e_x - \sum_{u \leq x \leq v} e_x \right) - \sum_{x \in P} a_x e_x$$

is a diagonal element in  $L_{uv}$ , and that

$$d + a = \left( \sum_{x \in P} e_x - \sum_{u \leq x \leq v} e_x \right) + \sum_{x < y} a_{xy} e_{xy} \in L_{uv}$$

satisfies condition (4.21). Thus  $\eta(d + a) \in L_{uv}$ , and with Lemma 4.23 also

$$\eta(a) = \eta(d + a) - \eta(d) = \eta(d + a) - d \in L_{uv}.$$

We infer that  $\eta(L_{uv}) \subseteq L_{uv}$  for all  $u \leq v \in P$ . □

We are all set to complete the proof of Theorem 4.4.

**Theorem 4.26.** *Let  $\eta \in \text{LAut}(FI(P))$  induce the identity map on  $FI(P)/Z$  with  $\eta(e_{xy}) = e_{xy}$  for all  $x \leq y \in P$ . Then  $\eta = \text{id}_{FI(P)}$ .*

*Proof.* Let  $a \in FI(P)$ . For any  $u \leq v \in P$  holds

$$a = \sum_{x \leq y} a_{xy} e_{xy} \in \sum_{x \in P} a_x e_x + \sum_{u \leq x \leq y \leq v} a_{xy} e_{xy} + L_{uv}.$$

With Lemma 4.25, we infer

$$\eta(a) \in \eta \left( \sum_{x \in P} a_x e_x \right) + \eta \left( \sum_{u \leq x \leq y \leq v} a_{xy} e_{xy} \right) + L_{uv}.$$

Note that the first sum describes a diagonal element, while the second sum is finite. Thus, with Lemma 4.23,

$$\eta(a) \in \sum_{x \in P} a_x e_x + \sum_{u \leq x \leq y \leq v} a_{xy} \eta(e_{xy}) + L_{uv} = \sum_{x \in P} a_x e_x + \sum_{u \leq x \leq y \leq v} a_{xy} e_{xy} + L_{uv}.$$

In particular,  $(\eta(a))_{uv} = a_{uv}$ . As this holds for all  $u \leq v \in P$ , we infer  $\eta(a) = a$ .  $\square$

#### REFERENCES

- [1] M. Brešar, P. Šemrl, *Mappings which preserve idempotents, local automorphisms, and local derivations*, Canad. J. Math. **45** (1993), 483–496.
- [2] M. Brešar, P. Šemrl, *On local automorphisms and mappings that preserve idempotents*, Studia Math. **113** (1995), 101–108.
- [3] R. Crist, *Local automorphisms*, Proc. Am. Math. Soc. **128** (1999), 1409–1414.
- [4] M. Dugas, B. Wagner, *Finitary incidence algebras and idealizations*, Linear and Multilinear Algebra **64** (2016), 1936–1951.
- [5] D. Hadwin, J. Li, *Local derivations and local automorphisms*, J. Math. Anal. Appl. **290** (2004), 702–714.
- [6] T. Jech, *Set Theory* (3rd Millenium Edition), Springer Monogr. Math., Springer, Berlin (2003).
- [7] N. S. Khripchenko, *Automorphisms of finitary incidence rings*, Algebra Discrete Math. **9** (2010), 78–97.
- [8] N. S. Khripchenko, B. V. Novikov, *Finitary incidence algebras*, Comm. Algebra **37** (2009), 1670–1676.
- [9] D. R. Larson, A. R. Sourour, *Local derivations and local automorphisms of  $\mathfrak{B}(X)$* , Proc. Symp. Pure Math. **51** (1990), 187–194.
- [10] G.-C. Rota, *On the foundations of combinatorial theory I: theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie **2** (1964), 340–368.
- [11] P. Šemrl, *Local automorphisms of standard operator algebras*, J. Math. Anal. Appl. **371** (2010), 403–406.
- [12] E. Spiegel, C. J. O’Donnell, *Incidence Algebras*, Monogr. Textbooks Pure Appl. Math. **206**, Marcel Dekker (1997).
- [13] R. P. Stanley, *Structure of incidence algebras and their automorphism groups*, Bull. Amer. Math. Soc. **76** (1970), 1236–1239.
- [14] R. P. Stanley, *Enumerative Combinatorics: Volume 1* (2nd Edition), Cambridge University Press, New York (2011).

(Jordan Courtemanche) DEPARTMENT OF MATHEMATICS, BAYLOR UNIVERSITY, ONE BEAR PLACE #97328, WACO, TX 76798-7328, USA

*E-mail address:* `Jordan.Courtemanche@baylor.edu`.

(Manfred Dugas) DEPARTMENT OF MATHEMATICS, BAYLOR UNIVERSITY, ONE BEAR PLACE #97328, WACO, TX 76798-7328, USA

*E-mail address:* `Manfred.Dugas@baylor.edu`

(Daniel Herden) DEPARTMENT OF MATHEMATICS, BAYLOR UNIVERSITY, ONE BEAR PLACE #97328, WACO, TX 76798-7328, USA

*E-mail address:* `Daniel.Herden@baylor.edu`